Polynomial formulations as a barrier for reduction-based hardness proofs

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Abstract

The Strong Exponential Time Hypothesis (SETH) asserts that for every $\varepsilon > 0$ there exists k such that k-SAT requires time $(2-\varepsilon)^n$. The field of fine-grained complexity has leveraged SETH to prove quite tight conditional lower bounds for dozens of problems in various domains and complexity classes, including Edit Distance, Graph Diameter, Hitting Set, Independent Set, and Orthogonal Vectors. Yet, it has been repeatedly asked in the literature whether SETH-hardness results can be proven for other fundamental problems such as Hamiltonian Path, Independent Set, Chromatic Number, MAX-k-SAT, and Set Cover.

In this paper, we show that fine-grained reductions implying even λ^n -hardness of these problems from SETH for $any \ \lambda > 1$, would imply new circuit lower bounds: super-linear lower bounds for Boolean series-parallel circuits or polynomial lower bounds for arithmetic circuits (each of which is a four-decade open question).

We also extend this barrier result to the class of parameterized problems. Namely, for every $\lambda > 1$ we conditionally rule out fine-grained reductions implying SETH-based lower bounds of λ^k for a number of problems parameterized by the solution size k.

Our main technical tool is a new concept called polynomial formulations. In particular, we show that many problems can be represented by relatively succinct low-degree polynomials, and that any problem with such a representation cannot be proven SETH-hard (without proving new circuit lower bounds).

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1 Introduction

In this paper, we explain the lack of hardness results based on the Strong Exponential Time Hypothesis for a large class of problems by proving that such hardness results would lead to new strong circuit lower bounds.

1.1 Background

The central question in complexity theory is to find the minimum time required to solve a given computational problem. Answering such a question involves proving lower bounds on computational complexity. Unconditional lower bounds remain elusive: for example, we do not know how to solve CNF-SAT in time $(2-\varepsilon)^n$ (where n is the number of variables in an input CNF formula and $\varepsilon > 0$ is a constant), and at the same time we have no tools to exclude the possibility of even an O(n) time algorithm. Super-linear lower bounds are only known for restricted models of computation.

For this reason, all existing lower bounds are conditional. Classical complexity theory, founded in the 1970's, considers polynomial-time reductions: we say that P reduces to Q and write $P \leq Q$, if a polynomial-time algorithm for Q can be used to solve P in polynomial time. Such a reduction may be viewed as a conditional lower bound: if the problem P cannot be solved in polynomial time, then neither can the problem Q. While polynomial-time reductions (conditionally) rule out polynomial-time algorithms for many problems, they say little about quantitative hardness of computational problems.

The recently developed field of fine-grained complexity aims to establish tighter connections between complexities of computational problems. By using fine-grained reductions, one can leverage algorithmic hardness assumptions to prove quantitative lower bounds for wide classes of problems. A fine-grained reduction, denoted $(P, p(n)) \leq (Q, q(n))$, implies that a faster than q(n)-time algorithm for Q leads to a faster than p(n)-time algorithm for P. The standard assumptions in this field (see Vassilevska Williams [Vas15, Vas18] for excellent surveys on this topic) are hardness of CNF-SAT [IP99, IPZ98], 3-SUM [GO95, Eri99], $Orthogonal\ Vectors$ [Wil05], $All\ Pairs\ Shortest\ Paths\ [VW10]$, $Online\ Matrix$ - $Vector\ Multiplication\ [HKNS15]$, and $Set\ Cover\ [CDL^+16]$.

One of the most popular fine-grained assumptions, the Strong Exponential Time Hypothesis (SETH), postulates that for every $\varepsilon > 0$ there exists a k such that k-SAT cannot be solved in time $(2 - \varepsilon)^n$. The Strong explanatory power of SETH is confirmed by many tight lower bounds for computational problems both in P and NP. We refer the reader to [Vas18, Section 3] for an extensive list of such results, and we list a few notable representatives below. The following upper bounds are known to be tight (up to small multiplicative factors) under SETH:

- n^2 for Orthogonal Vectors [Wil05] (where n is the number of vectors), 3/2-approximate Graph Diameter [RV13] (where n is the number of nodes in the input graph), and Edit Distance [BI15] (where n is the length of the input strings);
- 2^n for Hitting Set (where n is the size of the universe); and NAE-SAT (where n is the number of the variables) [CDL+16];
- n^k for k-Dominating Set [PW10] (where n is the number of nodes in the input graph and $k \geq 7$);
- 2^{tw} for *Independent Set* [LMS18] (where tw is the treewidth of the input graph).

Given such an extensive list of tight conditional lower bounds, one may speculate that SETH can explain many other current algorithmic barriers.

Open Problem 1. Can we prove λ^n -SETH hardness results for $\lambda > 1$ for any of the following problems: k-SAT (asked in [CDL⁺16, problem 5]), Hamiltonian Path (asked in [FK10, chapter 12]), Chromatic Number (asked in [LMS11, problem 5], [Zam21, problem 43], [CDL⁺16, problem 2]), Set Cover (asked in [CDL⁺16, problem 1]), Independent Set (asked in [LMS11, problem 4]), Clique, Vertex Cover, MAX-k-SAT, 3d-Matching?

Can we prove λ^k -SETH hardness results Section 2.6 for $\lambda > 1$ for any of the following parameterized problems (asked in [LMS11, problem 2]): k-Path, k-Vertex Cover, k-Tree, k-Steiner Tree, k-Internal Spanning Tree, k-Leaf Spanning Tree, k-Nonblocker, k-Path Contractibility, k-Cluster Editing, k-Set Splitting?

Barriers for hardness proofs. The first (and the only prior to this work) conditional barrier for proving SETH-hardness results was shown by Carmosino et al. [CGI+16]. For an integer k, k-TAUT is the language of all k-DNFs that are tautologies (note that a k-DNF formula ϕ is in k-TAUT if and only if the k-CNF formula resulting from negating ϕ is not in k-SAT). [CGI+16] defines a stronger version of SETH—Non-deterministic Strong Exponential Time Hypothesis (NSETH)—which postulates that for every $\varepsilon > 0$ there exists k such that even non-deterministic algorithms cannot solve k-TAUT on n-variate formulas in time $2^{(1-\varepsilon)n}$. While this conjecture is stronger than SETH, refuting even NSETH would imply strong lower bounds against Boolean series-parallel circuits [JMV15]. Carmosino et al. [CGI+16] proved that tight SETH-hardness results for 3-SUM, APSP, and some other problems would refute NSETH, and, thus, imply new circuit lower bounds (resolving a four-decade open question).

Circuit lower bounds. The barriers we show for SETH-hardness proofs also lie in the field of circuit complexity. Below we review two of the main challenges in this field, and we give rigorous definitions of the circuit models in Sections 2.1 and 2.2. The best known lower bound on the size of Boolean circuits computing functions in P is 3.1n - o(n) [LY22]. In fact, this bound remains the best known even for the much larger class of functions from E^{NP} even against the restricted model of series-parallel circuits. A long-standing open problem in Boolean circuit complexity is to find an explicit language that cannot be computed by linear-size circuits from various restricted circuit classes [Val77, AB09, Frontier 3].

Open Problem 2. Prove a lower bound of $\omega(n)$ on the size of Boolean series-parallel circuits computing a language from E^NP .

In contrast to the case of Boolean circuits, in the model of arithmetic circuits we have a superlinear lower bound of $\Omega(n \log n)$ [Str73a, BS83]. One of the biggest challenges in this area is to prove a stronger lower bound, for example, a lower bound of n^{γ} for a constant $\gamma > 1$.

Open Problem 3. For a constant $\gamma > 1$, prove a lower bound of n^{γ} on the arithmetic circuit complexity of a constant-degree polynomial that can be constructed in polynomial time $n^{O(1)}$.

1.2 Our Contribution

Despite much effort, we still do not have any SETH-hardness results for the problems listed in Open Problem 1. For example, an algorithm solving the Hamiltonian Path problem in time $2^n n^{O(1)}$

has been known for 60 years [Bel62, HK62], yet we don't have any improvements on the algorithm nor conditional lower bounds λ^n on the complexity of this problem. In this paper, we show that this barrier is no coincidence. Namely, we show that a resolution of Open Problem 1 would resolve Open Problem 2 or Open Problem 3. More specifically, any SETH-based exponential lower bound for Hamiltonian Path or any other problem from Open Problem 1 would imply a super-linear lower bound for series-parallel circuits or an arbitrarily large polynomial lower bound for arithmetic circuits.

Our first main result says that for a number of well-studied problems, any SETH lower bound of the form λ^n for a constant $\lambda > 1$ would imply new circuit lower bounds. (In the end of this section we clarify what we mean by SETH lower bounds, and our definition is quite general.)

Theorem 1.1. If at least one of the following problems

k-SAT, MAX-k-SAT, Hamiltonian Path, Graph Coloring, Set Cover, Independent Set, Clique, Vertex Cover, 3d-Matching

is λ^n -SETH-hard for a constant $\lambda > 1$, then at least one of the following circuit lower bounds holds:

- E^{NP} requires series-parallel Boolean circuits of size $\omega(n)$;
- for every constant $\gamma > 1$, there exists an explicit family of constant-degree polynomials over \mathbb{Z} that requires arithmetic circuits of size $\Omega(n^{\gamma})$.

While this result conditionally rules out, say, 1.1^n -hardness of Hamiltonian Path, it is still possible that the parameterized version of Hamiltonian Path, k-Path, is 1.1^k -hard for some function k := k(n). In our second result, we conditionally rule out even such hardness results (which might have seemed easier to obtain).

Theorem 1.2. If at least one of the following parameterized problems

k-Path, k-Vertex Cover, k-Tree, k-Steiner Tree, k-Internal Spanning Tree, k-Leaf Spanning Tree, k-Nonblocker, k-Path Contractibility, k-Cluster Editing, k-Set Splitting

is λ^k -SETH-hard for a constant $\lambda > 1$, then at least one of the following circuit lower bounds holds:

- E^{NP} requires series-parallel Boolean circuits of size $\omega(n)$;
- for every constant $\gamma > 1$, there exists an explicit family of constant-degree polynomials over \mathbb{Z} that requires arithmetic circuits of size $\Omega(n^{\gamma})$.

A couple of remarks about the main results of this work are in order. While an SETH-lower bound for any problem in the premise of Theorems 1.1 and 1.2 would imply that at least one of Open Problem 2 or Open Problem 3 must have an affirmative answer, it would not say which one!

The first part of the conclusion of Theorems 1.1 and 1.2 is really that "NSETH fails", and the series-parallel circuit lower bound comes as an already known consequence of that [JMV15].

¹The undirected version of Hamiltonian Path can be solved in time 1.66^n by Björklund's algorithm [Bjö10]. In this paper, we only consider the directed version of the Hamiltonian Path and k-Path problems. Since we're proving barriers for lower bounds, considering the harder directed version of a problem only makes our barrier results stronger.

On the definition of SETH-hardness. The Exponential Time Hypothesis (ETH), which follows from SETH [IPZ98], asserts that there is some non-explicit $\lambda > 1$ such that 3-SAT on n variables requires time λ^n . While SETH implies the existence of such $\lambda > 1$, it doesn't tell us anything about λ . In particular, it doesn't provide us with a lower bound on λ greater than 1. Each of the problems discussed above has a reduction from 3-SAT that preserves the size of the instance up to a constant factor [IPZ98, CFK⁺15, Theorem 14.6]. Thus, under SETH (or even ETH), none of these problems can be solved by algorithms running in time $(\lambda')^n$ for some non-explicit $\lambda' > 1$ which we cannot even bound away from 1.

The only currently known way to prove fine-grained and SETH-hardness results is via fine-grained reductions. Such a reduction gives a conditional lower bound of λ^n on the complexity of a computational problem for an *explicit* $\lambda > 1$. In this work, we (conditionally) rule out reduction-based proofs of SETH-hardness which we emphasize in the title of the paper.

To simplify the presentation, we follow the terminology of previous works in this area and say "a computational problem P is λ^n -SETH hard for $\lambda > 1$ " when we mean the following. There exists an *explicit* constant $\lambda > 1$ and a function $\delta := \delta(\varepsilon)$ such that for every k and $\varepsilon > 0$, there exists an algorithm A for k-SAT running in time $2^{(1-\delta(\varepsilon))n}$, making t calls to an oracle for P with input sizes n_1, \ldots, n_t satisfying

$$\sum_{i=1}^{t} \lambda^{(1-\varepsilon)n_i} \le 2^{(1-\delta(\varepsilon))n} .$$

In particular, this captures all known λ^n -SETH lower bounds in fine-grained complexity for explicit $\lambda > 1$.

On randomized reductions. While this work focuses on deterministic (i.e., non-randomized) reductions, now we discuss why randomized SETH-hardness reductions would also be surprising. If one of the problems under consideration is SETH-hard under deterministic reductions, then Theorems 1.1 and 1.2 imply that either NSETH is false (implying series-parallel circuit lower bounds) or we have high arithmetic circuit lower bounds. In fact, the same result holds for zero-error probabilistic reductions.

A randomized SETH-hardness reduction for any of the problems in the premises of Theorems 1.1 and 1.2 would either give us high arithmetic circuit lower bounds or a faster than 2^n two-round AM protocol for k-TAUT. While faster than 2^n two-round AM protocols are not known to imply circuit lower bounds, designing such a protocol for k-TAUT would still be a big achievement. The celebrated work of Williams [Wil16] gives a constant-round AM protocol running in time $2^{n/2}$ for k-TAUT.

Also, our result, together with the standard trick of simulating randomness by non-uniformity (see [CGI⁺16, Lemma 4]), gives us that a randomized SETH-hardness reduction for any of the problems in the premises of Theorems 1.1 and 1.2 implies either high arithmetic circuit lower bounds or that NUNSETH (Non-unifrom NSETH, see [CGI⁺16, Definition 4])) is false.

²The standard transformation of a constant-round AM protocol into a two-round AM protocol suffers a quadratic blow-up in size which results in a two-round protocol of trivial length 2^n .

1.3 Proof Overview

1.3.1 Non-parameterized Problems

We demonstrate our main result on the following example. Assume that the Hamiltonian Path problem is 1.5^n -SETH hard. We'll prove one of the two circuit lower bounds: a super-linear lower bound for Boolean series-parallel circuits computing a language from E^{NP} or an $n^{1.2}$ lower bound on the size of arithmetic circuits computing an explicit n-variate polynomial of constant degree. We note that the actual result proven in Theorem 1.1 rules out λ^n -hardness for every $\lambda > 1$ and is capable of proving arithmetic circuit lower bounds of n^{γ} for every $\gamma > 1$.

First we find a low-degree polynomial on exponentially-many variables that can be used to solve any Hamiltonian path instance of a given size. We call this a polynomial formulation of Hamiltonian Path.³ Then we show how to find a small arithmetic circuit computing this polynomial formulation. After that, we give a non-deterministic algorithm for the k-TAUT problem, and conclude with circuit lower bounds.

Polynomial formulations of Hamiltonian Path. First we give the following polynomial formulation of Hamiltonian Path on a graph with node set $[\ell]$. Without loss of generality, we assume that ℓ is a multiple of 10. For every set S of $\ell/10$ nodes, every node $u \in S$, and every node $v \notin S$, we introduce a variable $x_{S,u,v}$. Thus, in total we have $t \leq \binom{\ell}{\ell/10} \cdot \ell^2 \leq 2^{0.47\ell}$ variables. Consider the following degree-10 polynomial P of t variables. For each partition of $[\ell] = S_1 \sqcup \cdots \sqcup S_{10}$ into sets of size $\ell/10$, and for each $v_1 \in S_1, \ldots, v_{10} \in S_{10}$, and $v_{11} \in [\ell]$ we add the monomial

$$x_{S_1,v_1,v_2} \cdot x_{S_2,v_2,v_3} \cdots x_{S_{10},v_{10},v_{11}}$$
.

Now for an instance G of Hamiltonian path, where G is a graph on ℓ nodes, we assign the following values to the t variables of the polynomial P. If there is a Hamiltonian path visiting each node of S exactly once, starting at the node u and ending at a node neighboring v, then we set $x_{S,u,v}=1$, otherwise we set $x_{S,u,v}=0$. Note that the polynomial P evaluates to 0 if and only if the original graph has no Hamiltonian paths. Indeed, a Hamiltonian path can be partitioned into 10 parts of length $\ell/10$ with starting nodes v_1, \ldots, v_{10} such that all variables of the monomial $x_{S_1,v_1,v_2} \cdot x_{S_2,v_2,v_3} \cdots x_{S_{10},v_{10},v_{11}}$ are assigned ones. The converse is also true: if all variables of some monomial are assigned ones, then this gives us a Hamiltonian path in G.

Time complexity of polynomial formulations. Now we bound the running time for computing the polynomial P. This polynomial can be constructed by listing all 10-tuples of $(\ell/10)$ -subsets of $[\ell]$ together with nodes v_1, \ldots, v_{11} , which can be done in time $O\left(\binom{\ell}{\ell/10}^{10}\ell^{O(1)}\right) \leq 2^{4.7\ell}$. This polynomial is not particularly helpful for solving an instance of Hamiltonian Path on ℓ nodes since the problem can be solved in time $2^{\ell}\ell^{O(1)}$ while only constructing this polynomial takes time $2^{4.7\ell}$. Nevertheless, we'll later use the self-reducibility property of TAUT to reuse one polynomial to solve exponentially many instances of $Hamiltonian\ Path$.

Let us now bound the running time for computing the assignment of the variables for an instance G on ℓ nodes. In order to find the assignment of the variables, it's sufficient to solve t

³In fact, our main result (Theorem 4.3) rules out SETH-hardness results for all problems that admit polynomial formulations on c^n variables of instances of size n for every c > 1.

instances of Hamiltonian path on $\ell/10$ nodes. Since each instance can be solved in time $2^{\ell/10}\ell^{O(1)}$, we bound the total running time by

$$t \cdot 2^{\ell/10} \ell^{O(1)} \le \binom{\ell}{\ell/10} \cdot 2^{\ell/10} \cdot \ell^{O(1)} = O(2^{0.47\ell + \ell/10}) = O(1.49^{\ell})$$
.

Thus, in time $2^{4.7\ell}$ we construct a degree-10 polynomial with at most $t \leq 2^{0.47\ell}$ variables, such that each instance of *Hamiltonian path* on ℓ nodes can be reduced in time 1.49^{ℓ} to evaluating this polynomial at a 0/1 point. Note that since the polynomial has $2^{4.7\ell}$ monomials with coefficients one, and we only evaluate it at 0/1 points, the maximum value we can obtain is $\leq 2^{4.7\ell}$, so we can as well assume that our polynomial P is over \mathbb{Z}_p for $2 \cdot 2^{4.7\ell} \leq p \leq 4 \cdot 2^{4.7\ell}$.

The first circuit lower bound. If the constructed polynomial P of t variables doesn't have arithmetic circuits of size $t^{1.2}$, then we have our first circuit lower bound. Indeed, we have an explicit family of constant-degree t-variate polynomials P_t that require arithmetic circuits of size at least $t^{1.2}$. To see that this family of polynomials is explicit, recall that this t-variate polynomial can be constructed in time polynomial in the number of variables: $O\left(\binom{\ell}{\ell/10}^{10}\ell^{O(1)}\right) \leq t^{11}$. Thus, in the following we assume that the polynomial P does have circuits of size $t^{1.2}$. From this (together with the assumed 1.5^n -SETH hardness of $Hamiltonian\ Path$) we will prove the second circuit lower bound. In fact, we will refute NSETH, which, as discussed earlier, implies a super-linear circuit lower bound for Boolean series-parallel circuits [JMV15]. Therefore, in the rest of this section, we will show how to solve the k-TAUT problem in non-deterministic time $2^{(1-\varepsilon)n}$ for constant $\varepsilon > 0$.

Approximation of MACP. We have a t-variate polynomial P that solves $Hamiltonian\ Path$ on graphs with ℓ nodes, and we know that this polynomial has circuits of size $t^{1.2}$, but we don't have those circuits. We use the result of Strassen [Str73b] that asserts that if there is a circuit of size $t^{1.2}$ computing a degree-10 polynomial, then there exists a circuit of size $O(t^{1.2})$ computing the same polynomial, where every gate of the circuit computes a polynomial of degree at most 10 (see Corollary 3.4). We use the power of non-deterministic algorithm to guess such a circuit, but we need to verify that the guessed circuit indeed computes the polynomial P. For this, we write down the polynomials computed at each gate of the circuit. Since in our modified circuit, each gate computes a degree-10 polynomial, each such polynomial can be written in time $O(t^{10})$. Performing all operations (modulo p) over the polynomials in this circuit will take time $O(t^{1.2} \cdot t^{20} \cdot \log^2 p) \le 2^{21.2 \cdot 0.47\ell} \cdot \ell^{O(1)} \le 2^{10\ell}$. By comparing the list of the monomials of the polynomial P with the monomials computed at the output gate, we verify the guessed circuit.

Non-deterministic algorithm for k-TAUT. Below we present an algorithm solving every k-TAUT instance on n variables in non-deterministic time $2^{(1-\varepsilon)n}$ for constant ε independent of k. Recall that this refutes NSETH and, thus, implies the second circuit lower bound.

We consider all assignments of the first $(1-\alpha)n$ Boolean variables for $\alpha=1/20$. This gives us a set of $2^{(1-\alpha)n}$ instances of k-TAUT on αn variables each. Solutions to these $2^{(1-\alpha)n}$ instances will give us the solution to the original instance. Each of these instances of k-TAUT will later be reduced (via the assumed fine-grained reduction from k-SAT to Hamiltonian Path) to (possibly exponentially many) instances of Hamiltonian Path. Since we assume 1.5^n -SETH hardness of Hamiltonian Path, an instance of k-TAUT on αn variables must be reduced to instances of Hamiltonian Path on at most ℓ nodes, where ℓ satisfies $1.5^{\ell} \leq 2^{\alpha n}$. In particular, $\ell < n/11$. In this proof

overview we assume for simplicity that all instances produced by the fine-grained reduction have size ℓ , (in the full proof of Theorem 4.3 we generate ℓ different polynomials, one for each integer up to ℓ , to solve *Hamiltonian Path* instances of any size $\leq \ell$).

First, we construct the polynomial P that solves the instances of $Hamiltonian\ Path$ on ℓ nodes. Recall that we can construct this polynomial in time $2^{4.7\ell} \ll 1.9^n$ (we only need to write down this polynomial once, and then we will use this polynomial for all instances of $Hamiltonian\ Path$). Then we find a circuit of size $O(t^{1.2})$ computing P in non-deterministic time $2^{10\ell} \ll 1.9^n$. Now we are ready to solve all $2^{(1-\alpha)n}$ instances of k-TAUT on αn variables. Indeed, each such instance we reduce to (a number of instances of) $Hamiltonian\ Path$ on ℓ nodes. We find the assignment of the variables of P in time 1.49^{ℓ} , and evaluate the circuit of size $O(t^{1.2})$ modulo p in time $O(t^{1.2}\log^2 p) \le 1.49^{\ell}$. Since we solve each instance of $Hamiltonian\ Path$ on ℓ nodes in time 1.49^{ℓ} , 1.5^n -SETH hardness of $Hamiltonian\ Path$ implies a $2^{(1-\delta)\alpha n}$ -algorithm for k-TAUT on αn variables (for constant $\alpha > 0$ independent of k). Therefore, each of the $2^{(1-\alpha)n}$ instances of k-TAUT is solved in time $2^{(1-\delta)\alpha n}$, and the total (non-deterministic) running time to solve the original instance of k-TAUT is $2^{(1-\delta)\alpha n} \cdot 2^{(1-\delta)\alpha n} = 2^{(1-\alpha\delta)n}$.

Summary. We start by showing that $Hamiltonian\ Path\ can$ be expressed as a constant-degree polynomial P in exponentially many variables. That is, checking whether the input graph has a Hamiltonian path boils down to evaluating P.

Assuming that P can be computed by a small arithmetic circuit C (if this is not the case, we have an arithmetic circuit lower bound), we show that there exists a homogeneous circuit C' computing P that is not much larger than C. We guess C' non-deterministically. The fact that C' is homogeneous and that P has constant degree allows us to efficiently verify the correctness of C' by checking the computation at each gate manually.

The circuit C' allows us to solve Hamiltonian Path efficiently. Since we have a fine-grained reduction from k-SAT to Hamiltonian Path, this implies a fast non-deterministic algorithm for k-TAUT. The obtained algorithm refutes NSETH, which, in turn, gives super-linear lower bounds for Boolean series-parallel circuits.

1.3.2 Parameterized Problems

In the (parameterized) k-Path problem, the goal is to check if the given graph on n nodes contains a simple path on k nodes. Similarly to the case of Hamiltonian Path, the best known algorithm for k-Path runs in time 2^k [Wil09]. The result sketched in the previous section (conditionally) rules out λ^n -SETH hardness of Hamiltonian Path for every $\lambda > 1$. Yet, this leaves a possibility to prove a 2^k lower bound on the complexity of k-Path for some function k := k(n). We extend our framework to parameterized problems to prove the same barriers for SETH-hardness for parameterized problems.

While most of the machinery from the previous section extends to the case of parameterized problems, the new issue arises in the polynomial formulations of the problems. Indeed, if we apply the polynomial formulation from the previous section to the k-Path problem, then we inevitably have at least $\binom{n}{k/10} \ge n^{\Omega(k)}$ variables. Such a polynomial formulation can only rule out $\binom{n^{\Omega(k)}n^{O(1)}}{n^{O(1)}}$. SETH hardness results, which are of no interest since the problem can be solved in time $2^k n^{O(1)}$. To overcome this issue, we design a different polynomial formulation of the parameterized k-Path problem. This formulation uses a certain pseudorandom object called a splitter, on which we elaborate below.

Splitters. A family H of functions $f:[n] \to [k]$ is a k-perfect hash family if for every $S \subseteq [n]$ of size |S| = k, there is a function $h \in H$ which is injective on S. An (n, k, ck)-splitter is a relaxation of this notion where the functions f have range [ck] for a constant $c \ge 1$. There are known constructions of splitters [NSS95] and k-perfect hash families [AYZ95] of size $e^{k(1+o(1))}k\log n$, but our polynomial formulations require splitters of size $e^{\frac{k}{g(c)}}$ for an unbounded function g. While a simple probabilistic argument shows that a set of $\approx e^{\frac{k}{c}} \cdot k\log n$ random functions forms an (n,k,ck)-splitter with high probability, in Section 5, we show how to efficiently construct an explicit family of such functions of size $e^{\frac{k}{c}(1+o(1))} \cdot k\log n$.

Polynomial formulations. The color-coding technique, introduced in [AYZ95], solves k-Path on a graph with node set [n] as follows: assign a random color $c \in [k]$ to every node; then, all nodes of a k-path receive different colors with probability about e^{-k} ; at the same time, one can find such a colorful path in time $2^k n^{O(1)}$. This gives us a randomized $2^{O(k)} n^{O(1)}$ -time algorithm for k-Path. This algorithm can be derandomized by utilizing a k-perfect family H of hash functions $f:[n] \to [k]$: go through all $f \in H$ and assign the color f(v) to every node v [AYZ95]. Since H guarantees that for every k-path there is a coloring $f \in H$ that assigns different colors to all nodes of the path, one of the hash functions f will lead to a k-path. [AYZ95] gives a construction of H of size $e^{k(1+o(1))}$, but this would result in a polynomial formulation with $t \ge e^{k(1+o(1))}$ variables. Recall that in order to prove a lower bound of t^{γ} on the size of arithmetic circuits, we need t^{γ} to be much less than the assumed complexity of k-Path. Thus, for our purposes, we need a family of hash function of much smaller size. To achieve this, we allow a larger number of colors. This will decrease the number of variables as desired at the cost of increasing the degree and the time needed to compute the coefficients of the polynomial. Fortunately, our construction is robust enough to tolerate this drawback.

We take a family H of (n, k, ck)-splitters of size $e^{\frac{k}{c}}$ that can be computed in time $2^{9k}n^{O(1)}$ described above. Given a coloring $f: [n] \to [ck]$, we find an f-colorful k-path in time $2^{ck}n^{O(1)}$ [AYZ95].

The main idea of the polynomial formulation then is the following. For a coloring $f:[n] \to [ck]$, an f-colorful k-path π can be partitioned into c paths π_1, \ldots, π_c such that each path uses at most k/c colors. Then, a partition π_1, \ldots, π_c is valid if the paths are color-disjoint and there is an edge from the last node of π_i to the first node of π_{i+1} , for all $i \in [c-1]$. Such valid paths will lead to monomials of degree c, and the resulting polynomial will have only $e^{\frac{k}{c}}n^{O(1)}$ variables for arbitrarily large constant c.

1.4 Related Work

The closest in spirit to our result is the work of Carmosino et al. [CGI⁺16]. The work [CGI⁺16] proves that if APSP, 3-SUM, or one of a few other problems is SETH-hard, then NSETH is false. In this paper, we relax the implication but prove a stronger barrier for a much wider class of problems: if one of the problems listed in Open Problem 1 is SETH-hard, then NSETH is false or we have arbitrarily large polynomial lower bounds for arithmetic problems. Another important difference between [CGI⁺16] and the present work is that [CGI⁺16] rules out *some* hardness results (say, $n^{3/2}$ -SETH hardness of 3-SUM), while the present work rules out *all* exponential SETH-hardness results λ^n for an explicit constant $\lambda > 1$.

The work of Kabanets and Impagliazzo [KI03] is the closest one to our main result in terms of techniques. In fact, we view our main result (Theorem 4.3) as a fine-grained version of [KI03] with

the following modifications. The work [KI03] shows that if $Polynomial\ Identity\ Testing\ (PIT)$ has a deterministic polynomial time algorithm, then either NEXP $\not\subseteq P/poly$, or the Permanent problem requires super-polynomial size arithmetic circuits. To show this, they guess small arithmetic circuits and verify them using the assumed deterministic algorithm for PIT and downward self-reducibility of Permanent. In this paper, efficient verification is possible due to the low degree of the polynomial computed by the circuit and the homogenization trick allowing us to consider only circuits with a specific structure.

Finally, several works provided efficient algorithms for problems studied in fine-grained complexity (e.g., Grønlund and Pettie [GP18] for 3-SUM and Williams [Wil16] for Multipoint Circuit Evaluation) using self-reducibility of the problem and a construction that improves on the trivial running time for small instances of the problem. In this paper, we also use the self-reducibility trick: first, in the allocated time we construct arithmetic circuits solving all instances of k-SAT on αn variables significantly faster than in $2^{\alpha n}$, and then we reduce an n-variate instance of k-SAT to a series of instances on αn variables for a sufficiently small $\alpha > 0$.

1.5 Structure

The rest of the paper is organized as follows. In Section 2, we give all necessary background material, including the definitions of SETH-hardness and polynomial formulations. In Section 3, we define an approximate version of the Minimum Arithmetic Circuit Problem (MACP), and provide a non-deterministic algorithm for this version of the problem. In Section 4, we prove the main result of this paper: SETH-hardness of a problem admitting polynomial formulations would imply one of the two aforementioned circuit lower bounds. Section 5 contains a construction of a pseudorandom object needed for certain polynomial formulations: deterministic splitters over alphabets of linear size. In Section 6, we give polynomial formulations of all problems considered in this paper. Finally, Appendix A contains proofs of technical claims omitted in the main part of the paper.

2 Preliminaries

For two sets S and T, by $S \sqcup T$ we denote their disjoint union. All logarithms in this paper are base 2, i.e., $\log 2^n = n$. Recall that the $O^*(\cdot)$ notation suppresses polynomial factors, e.g., $n^2 2^n = O^*(2^n)$.

We use square brackets in two ways: for a positive integer p, $[p] = \{1, ..., p\}$; for a predicate P, [P] = 1 if P is true and [P] = 0 otherwise (this is Iverson bracket).

2.1 Boolean Circuits

Definition 2.1. A Boolean circuit C with variables x_1, \ldots, x_n is a directed acyclic graph as follows. Every node has in-degree zero or two. The in-degree zero nodes are labeled either by variables x_i or constants 0 or 1. The in-degree two nodes are labeled by binary Boolean functions that map $\{0,1\}^2$ to $\{0,1\}$. The only gate of out-degree zero is the output of the circuit.

A Boolean circuit C with variables x_1, \ldots, x_n computes a Boolean function $f: \{0,1\}^n \to \{0,1\}$ in a natural way. We define the size of C as the number of gates in it, and the Boolean circuit complexity of a function as the minimum size of a circuit computing it.

A circuit is called *series-parallel* if there exists a numbering ℓ of the circuit's nodes such that for every wire (u, v), $\ell(u) < \ell(v)$, and no pair of wires (u, v), (u', v') satisfies $\ell(u) < \ell(u') < \ell(v) < \ell(v')$.

The best known lower bound on the size of a Boolean circuit for functions in P is 3.1n - o(n) [LY22]. In fact, this bound remains the best known bound even for the much larger class of languages E^{NP} even against the restricted model of series-parallel circuits. A long-standing open problem in Boolean circuit complexity is to find an explicit language that cannot be computed by linear-size circuits from various restricted circuit classes [Val77, AB09, Frontier 3].

Open Problem 2. Prove a lower bound of $\omega(n)$ on the size of Boolean series-parallel circuits computing a language from E^NP .

2.2 Arithmetic Circuits

Definition 2.2. An arithmetic circuit C over a ring R and variables x_1, \ldots, x_n is a directed acyclic graph as follows. Every node has in-degree zero or two. The in-degree zero nodes are labeled either by variables x_i or elements of R. The in-degree two nodes are labeled by either + or \times . Every gate of out-degree zero is called an output gate.

We will typically take R to be \mathbb{Z} or \mathbb{Z}_p for a prime number p. A single-output arithmetic circuit C over R computes a polynomial over R in a natural way. We say that C computes a polynomial $P(x_1, \ldots, x_n)$ if the two polynomials are *identical* (as opposed to saying that C computes P if the two polynomials agree on every assignments of $(x_1, \ldots, x_n) \in R^n$). We define the size of C as the number of edges in it, and the arithmetic circuit complexity of a polynomial as the minimum size of a circuit computing it.

While it's known [Str73a, BS83] that the polynomial $x_1^r + \ldots + x_n^r$ requires arithmetic circuits over \mathbb{F} of size $\Omega(n \log(r))$ (if r doesn't divide the characteristic of \mathbb{F}), one of the biggest challenges in algebraic complexity is to prove stronger lower bounds on the arithmetic circuit complexity of an explicit polynomial of constant degree.

Open Problem 3. For a constant $\gamma > 1$, prove a lower bound of n^{γ} on the arithmetic circuit complexity of a constant-degree polynomial that can be constructed in polynomial time $n^{O(1)}$.

2.3 SETH Conjectures

Below, we state rigorously two SETH-conjectures that we will use in this work.

• Strong exponential time hypothesis (SETH) [IPZ98, IP99]: for every $\varepsilon > 0$, there exists k such that

$$k$$
- $SAT \notin \mathsf{TIME}[2^{(1-\varepsilon)n}]$.

• Non-deterministic SETH (NSETH) [CGI+16]: for every $\varepsilon > 0$, there exists k such that

$$k$$
- $TAUT \notin \mathsf{NTIME}[2^{(1-\varepsilon)n}]$,

where k-TAUT is the language of all k-DNFs that are tautologies.

[JMV15] proved that if SETH is false, then E^NP requires series-parallel Boolean circuits of size $\omega(n)$. [CGI+16, Corollary B.3.] extended this result and showed that refuting NSETH is sufficient for such a circuit lower bound.

Theorem 2.3 ([CGI⁺16, Corollary B.3.]). If NSETH is false then E^{NP} requires series-parallel Boolean circuits of size $\omega(n)$.

2.4 Fine-grained Reductions

In this section we rigorously define fine-grained reductions and SETH-hardness. First, we give the definition of fine-grained reductions from [Vas15, Definition 6] with a small modification that specifies the dependence of δ on ε (we'll need this modification when defining SETH-hardness as SETH-hardness is a *sequence* of reductions from k-SAT for every value of k).

Definition 2.4 (Fine-grained reductions). Let P, Q be problems, $p, q: \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ be non-decreasing functions and $\delta: \mathbb{R}_{>0} \to \mathbb{R}_{>0}$. We say that (P, p(n)) δ -fine-grained reduces to (Q, q(n)) and write $(P, p(n)) \leq_{\delta} (Q, q(n))$, if for every $\varepsilon > 0$ and $\delta = \delta(\varepsilon) > 0$, there exists an algorithm \mathcal{A} for P with oracle access to Q, a constant d, a function $t(n): \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$, such that on any instance of P of size n, the algorithm \mathcal{A}

- runs in time at most $d(p(n))^{1-\delta}$;
- produces at most t(n) instances of Q adaptively: every instance depends on the previously produced instances as well as their answers of the oracle for Q;
- the sizes n_i of the produced instances satisfy the inequality

$$\sum_{i=1}^{t(n)} q(n_i)^{1-\varepsilon} \le d(p(n))^{1-\delta}.$$

We say that (P, p(n)) fine-grained reduces to (Q, q(n)) and write $(P, p(n)) \leq (Q, q(n))$ if $(P, p(n)) \leq_{\delta} (Q, q(n))$ for some function $\delta \colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$.

It is not difficult to see that if $(P, p(n)) \leq (Q, q(n))$, then any improvement over the running time q(n) for the problem Q implies an improvement over the running time p(n) for the problem P: for any $\varepsilon > 0$, there is $\delta > 0$, such that if Q can be solved in time $O(q(n)^{1-\varepsilon})$, then P can be solved in time $O(p(n)^{1-\delta})$.

Definition 2.5 (SETH-hardness). For a constant $\lambda > 1$, we say that a problem P is λ^n -SETH-hard if there exists a function $\delta \colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ and for every $k \in \mathbb{N}$,

$$(k\text{-}SAT, 2^n) \leq_{\delta} (P, \lambda^n)$$
.

If a problem P is λ^n -SETH-hard, then any algorithm solving P in time $\lambda^{(1-\varepsilon)n}$ implies an algorithm solving k-SAT in time $2^{(1-\delta(\varepsilon))n}$ for all k, thus, breaking SETH.

We now similarly define SETH-hardness of parameterized problems.

Definition 2.6 (SETH-hardness of parameterized problems). Let P be a parameterized problem with a parameter k, $\lambda > 1$ be a constant, and $\delta \colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$. We say that P is λ^k -SETH-hard if for every $q, d \in \mathbb{N}$, $\varepsilon > 0$, and $\delta = \delta(\varepsilon) > 0$, there exists an algorithm \mathcal{A} for q-SAT with oracle access to P, a function $t(n) \colon \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$, such that on any instance of q-SAT of size n, the algorithm \mathcal{A}

- runs in time at most $O(2^{(1-\delta)n})$;
- produces at most t(n) instances of P adaptively: every instance depends on the previously produced instances as well as their answers of the oracle for P;

• the length ℓ_i and parameters k_i of the produced instances satisfy the inequality

$$\sum_{i=1}^{t(n)} \lambda^{(1-\varepsilon)k_i} \cdot \ell_i^d \le O(2^{(1-\delta)n}).$$

It particular, if a parameterized problem P is λ^k -SETH-hard, then any algorithm solving P in time $\lambda^{(1-\varepsilon)k}|x|^d$ implies an algorithm solving q-SAT in time $O(2^{(1-\delta(\varepsilon))n})$ for all q, thus, breaking SETH.

2.5 Polynomial Formulations

In this work, we consider polynomials over a ring R, where R is typically \mathbb{Z} or \mathbb{Z}_p for a prime number p. By a family of polynomials \mathcal{P} we mean an infinite sequence of polynomials P_{i_1}, P_{i_2}, \ldots , such that $i_1 < i_2 < \cdots$ and P_n is a multivariate polynomial depending on n variables. We say that \mathcal{P} has degree d(n) if, for every n, every monomial of P_n has total degree at most d(n).

Definition 2.7 (Δ -explicit family of polynomials). For a constant Δ , we say that \mathcal{P} is Δ -explicit, if, for all n, the degree of P_n is at most Δ and all coefficients of P_n can be computed (simultaneously) in time $O(n^{\Delta})$.

Equipped with this definition, we're in a position to state the required properties of polynomial formulations that will allow us to prove barriers for hardness proofs. For the case of non-parameterized problems, we define polynomial formulations as follows.

Let A be a computational problem, and for every $n \in \mathbb{N}$, let I_n be the set of instances of A of size n. A polynomial formulation of a computational problem A is a Δ -explicit family of polynomials $\mathcal{P} = (P_s)_{s \geq 1}$ and a family of maps $\phi = (\phi_n)_{n \geq 1}$ where $\phi_n \colon I_n \to \mathbb{Z}^{s(n)}$ satisfying the following. In order to check if $x \in I_n$ is a yes instance of A, it suffices to map $y = \phi(x) \in \mathbb{Z}^{s(n)}$ and evaluate the corresponding polynomial $P_{s(n)}(y)$.

Definition 2.8 (Polynomial formulations). Let A be a computational problem and for every $n \in \mathbb{N}$, let I_n be the set of instances of A of size n. Let Δ be a constant, $T: \mathbb{N} \to \mathbb{N}$ be a time bound, and $\mathcal{P} = (P_1, P_2, \dots)$ be a Δ -explicit family of polynomials over \mathbb{Z} . We say that \mathcal{P} is a Δ -polynomial formulation of A of complexity T, if there exist

- a non-decreasing function $s: \mathbb{N} \to \mathbb{N}$ satisfying $s(n) \leq T(n)$, and an algorithm computing s(n) in time T(n);
- a family of mappings $\phi = (\phi_1, \phi_2, ...)$, where $\phi_n \colon I_n \to \mathbb{Z}^{s(n)}$, and an algorithm evaluating ϕ_n at any point in time T(n)

such that the following holds. For every $n \in \mathbb{N}$ and every $x \in I_n$,

- $P_{s(n)}(\phi_n(x)) \neq 0 \Leftrightarrow x \text{ is a yes instance of } A;$
- $|P_{s(n)}(\phi_n(x))| < 2^{s(n)}$.

In order to define polynomial formulations of parameterized problems, we need to make the following changes. An instance of the problem is now a pair $(x,k) \in I_n \times \mathbb{N}$, where k is the parameter. The function s now depends on x and the value of k. Similarly, each ϕ_n now also

depends on k. The time bounds on the evaluation of s and ϕ are now $T(k)|x|^{O(1)}$ rather than T(n). Since the size n of the instance x doesn't appear in these time bounds anymore (it's now replaced by the length of the bit representation |x| of x), we don't need the index n in ϕ_n , and we merge the functions $(\phi_1, \phi_2, ...)$ into one function ϕ .

Definition 2.9 (Polynomial formulations of parameterized problems). Let A be a parameterized computational problem and let $\mathcal{I} \times \mathbb{N}$ be the set of all instances of A, where for an instance $(x, k) \in \mathcal{I} \times \mathcal{N}$, k is the value of the parameter. Let Δ be a constant, $T : \mathbb{N} \to \mathbb{N}$ be a time bound, and $\mathcal{P} = (P_1, P_2, \dots)$ be a Δ -explicit family of polynomials over \mathbb{Z} . We say that \mathcal{P} is a Δ -polynomial formulation of A of complexity T, if there exist

- a function $s: \mathcal{I} \times \mathbb{N} \to \mathbb{N}$ satisfying $s(x,k) \leq T(k)|x|^{\Delta}$, and an algorithm computing s(x,k) in time $T(k)|x|^{\Delta}$;
- a function $\phi \colon \mathcal{I} \times \mathbb{N} \to \mathbb{Z}^*$ such that $\phi(x,k) \in \mathbb{Z}^{s(x,k)}$, and an algorithm computing $\phi(x,k)$ in time $T(k)|x|^{\Delta}$

such that the following holds. For every $(x,k) \in \mathcal{I} \times \mathbb{N}$,

- $P_{s(x,k)}(\phi(x,k)) \neq 0 \Leftrightarrow (x,k)$ is a yes instance of A;
- $|P_{s(x,k)}(\phi(x,k))| < 2^{T(k)|x|^{\Delta}}$.

2.6 Computational Problems

In this work, we show barriers to proving hardness for the following non-parameterized and parameterized problems.

Non-parameterized problems. For each problem below, the specified parameter n is used as the default size measure when bounding the complexity of the problem. It is well known that each of these problems can be solved in time $2^{O(n)}$.

- k-SAT: given a formula F in k-CNF over n variables, check if F has a satisfying assignment.
- MAX-k-SAT: given a formula F in k-CNF over n variables and an integer t, check if it is possible to satisfy at least t clauses of F.
- Hamiltonian Path: given a directed graph G with n nodes, check whether G contains a cycle visiting every node exactly once.
- Graph Coloring: given a graph G with n nodes and an integer t, check whether G can be colored properly using at most t colors.
- Set Cover: given a set family $\mathcal{F} \subseteq 2^{[n]}$ of size $n^{O(1)}$ and an integer t, check whether one can cover [n] with at most t sets from \mathcal{F} .
- Independent Set: given a graph G with n nodes and an integer t, check whether G contains an independent set of size at least t.
- Clique: given a graph G with n nodes and an integer t, check whether G contains a clique of size at least t.

- Vertex Cover: given a graph G with n nodes and an integer t, check whether G contains a vertex cover of size at most t.
- 3d-Matching: given a 3-uniform 3-partite hypergraph G with parts of size n and an integer t, check whether G contains a matching of size at least t.

Parameterized problems. Each of the problems below comes with a parameter k and we are interested to know how the complexity of the problem grows as a function of the input length and k. We say that a problem with a parameter k belongs to the class FPT if it can be solved in time $O^*(f(k))$ for some computable function f. Similarly to the case of non-parameterized problems, we denote the size of an instance x of a problem (the number of nodes in the input graph, the number of variables in the input formula) by n, and we denote the length (the length of the binary representation of the instance x) by |x|.

- k-Path: given a graph G, check whether G contains a simple path with k nodes.
- k-Vertex Cover: given a graph G, check whether G contains a vertex cover of size at most k.
- k-Tree: given a graph G and a tree T with k nodes, check whether there exists a (not necessarily induced) copy of T in G.
- k-Steiner Tree: given a graph G(V, E) with (integer non-negative) edge weights and a subset $T \subseteq V$ of its nodes of size k, and an integer $0 \le t \le |V|^{O(1)}$, check whether there is a tree in G of weight at most t containing all nodes from T.
- k-Internal Spanning Tree: given a graph G, check whether there is a spanning tree of G with at least k internal nodes.
- k-Leaf Spanning Tree: given a graph G, check whether there is a spanning tree of G with at least k leaves.
- k-Nonblocker: given a graph G, check whether G contains a subset of nodes of size at least k whose complement is a dominating set in G.
- k-Path Contractibility: given a graph G, check whether it is possible to contract at most k edges in G to turn it into a path.
- k-Cluster Editing: given a graph G, check whether it is possible to turn G into a cluster graph (a set of disjoint cliques) using at most k edge modifications (additions and deletions).
- k-Set Splitting: given a set family $\mathcal{F} \subseteq 2^{[n]}$ of size $n^{O(1)}$, check whether there exists a partition of [n] into two sets that splits at least k sets from \mathcal{F} .

3 Minimum Arithmetic Circuit Problem

In this section, we show that for polynomials of constant degree one can find arithmetic circuits of size close to optimal in nondeterministic polynomial time.

Definition 3.1. Let $\mathcal{P} = (P_1, P_2, \dots)$ be a family of polynomials. The minimum arithmetic circuit problem for \mathcal{P} , denoted by $MACP_{\mathcal{P}}(n,s)$, is: given $n,s \in \mathbb{N}$, find an arithmetic circuit of size at most s computing P_n , or report that there is no such circuit.

It is known that when \mathcal{P} is the family of permanent polynomials, $MACP_{\mathcal{P}}$ can be solved in time $(ns)^{O(1)}$ either by an MA-protocol or by a nondeterministic Turing machine with an oracle access to the polynomial identity testing problem (PIT) [KI03]. In our setting, we do not have oracle access to PIT nor do we have any randomness. To make up for that, we consider the following approximate version of MACP.

Definition 3.2. Let p be a prime number, $\mathcal{P} = (P_1, P_2, \dots)$ be a family of polynomials over \mathbb{Z}_p , and $c \geq 1$ be an integer parameter. The problem Gap-MACP_{\mathcal{P},c,p}(n,s) is: given $n,s \in \mathbb{N}$, output an arithmetic circuit over \mathbb{Z}_p of size at most cs computing P_n , if P_n can be computed in \mathbb{Z}_p by a circuit of size at most s; output anything otherwise.

The reason we allow an abitrary output in case P_n does not have a circuit of size s is the following. Right after solving the Gap-MACP problem, we will verify that the found circuit is correct. Thus, if instead of a circuit of size cs computing P_n , we are given a circuit that doesn't compute P_n correctly, we will reject it in the verification stage. The parameter p here is needed to have control over the maximum value of coefficients when expanding the guessed circuit as a polynomial.

We will use the following result proven by Strassen [Str73b] (see also [BCS97, Chapter 7.1] or [SY10, Theorem 2.2]). Recall that a polynomial is *homogeneous* if all its monomials have the same degree. We say that a circuit is homogeneous if all its gates compute homogeneous polynomials. For a polynomial P, the homogeneous part of P of degree i is the sum of all monomials of P of degree exactly i.

Theorem 3.3 ([Str73b]). There exists a constant $\mu' > 0$ such that the following holds. If a degree- Δ polynomial P can be computed by an arithmetic circuit of size s, then there exists a homogeneous circuit C' of size at most $\mu'\Delta^2s$ computing P such that the $\Delta + 1$ outputs of C' compute the homogeneous parts of P.

We use Theorem 3.3 to conclude that at the expense of increasing the circuit size by a factor of $O(\Delta^2)$, we can assume that an arithmetic circuit computing a degree- Δ polynomial contains only gates computing polynomials of degree at most Δ .

Corollary 3.4. There exists a constant $\mu > 0$ such that the following holds. If a degree- Δ polynomial P can be computed by an arithmetic circuit of size s, then P can be computed by a (single-output) circuit C of size at most $\mu \Delta^2 s$ such that all gates of C compute polynomials of degree at most Δ .

Proof. In order to construct the circuit C we take the homogeneous circuit C' guaranteed by Theorem 3.3, remove all gates computing polynomials of degree greater than Δ , and sum up all $\Delta + 1$ output gates of C' in the output of C. Since sums and products of degree- $(\Delta + 1)$ homogeneous polynomials can't compute non-trivial polynomials of degree $\leq \Delta$, removing gates computing polynomials of degree greater than Δ doesn't affect the output gates of C'. Since the outputs of C' compute the homogeneous parts of P, the output of C computes P, which finishes the proof of the corollary.

We now prove that for polynomials of bounded degree, Gap-MACP can be solved in non-deterministic polynomial time.

Lemma 3.5. There exists a constant $\mu > 0$ such that for every Δ -explicit family of polynomials \mathcal{P} and every prime number p,

$$\operatorname{Gap-MACP}_{\mathcal{P},\mu\Delta^2,p}(n,s) \in \mathsf{NTIME}[O(\Delta^2 s n^{2\Delta} \log^2 p)]\,.$$

Proof. We present a non-deterministic algorithm that, given a polynomial P of circuit complexity s, finds a circuit of size at most cs for $c = \mu \Delta^2$.

First we note that Corollary 3.4 guarantees the existence of a circuit C over \mathbb{Z}_p of size cs computing P such that each gate of C computes a polynomial of degree at most Δ .

We non-deterministically guess such a circuit C, and verify if it computes P correctly. If it does, we output C, and we output an empty circuit otherwise. It remains to show that in the specified time we can verify that C computes P. To do this, we start with the circuit inputs and proceed to its output, and write down the polynomial over \mathbb{Z}_p computed by each gate as a sum or product of polynomials of its input gates. There are at most $2n^{\Delta}$ monomials in a polynomial of degree at most Δ . Computing sums and products of such polynomials boils down to at most $O(n^{2\Delta})$ arithmetic operations with their coefficients. As every coefficient of a polynomial over \mathbb{Z}_p is specified by $\log p$ bits, any such arithmetic operation takes time $O(\log^2 p)$. Putting it all together, we expand each of the $\mu\Delta^2 s$ gates, expanding each gate takes $O(n^{2\Delta})$ arithmetic operations, each arithmetic operation takes time $O(\log^2 p)$. Thus, the total time of expanding C in \mathbb{Z}_p is

$$O(\mu \Delta^2 s \cdot n^{2\Delta} \cdot \log^2 p)$$
.

To nondeterministically solve Gap-MACP_{$\mathcal{P},\mu\Delta^2,p$}(n,s), we guess C and expand it in \mathbb{Z}_p as discussed above. Since P_n is from a Δ -explicit family \mathcal{P} , it can be written as a sum of monomials in time $O(n^{\Delta})$ (recall Definition 2.7). Then, it remains to compare the coefficients of the two sequences of monomials.

4 Main Results

In this section, we state the main results of this paper. First, we state Lemmas 4.1 and 4.2 asserting that every problem defined in Section 2.6 admits polynomial formulations, we'll prove these lemmas in Sections 6.1 and 6.2. Then, in Sections 4.1 and 4.2 we prove barriers to proving hardness results for non-parameterized and parameterized problems admitting polynomial formulations. Finally, we conclude that such barriers hold for for all problems from Section 2.6.

Lemma 4.1. For every c > 1, there is $\Delta = \Delta(c)$, such that each of the following problems

k-SAT, MAX-k-SAT, Hamiltonian Path, Graph Coloring, Set Cover, Independent Set, Clique, Vertex Cover, 3d-Matching

admits a Δ -polynomial formulation of complexity c^n .

Lemma 4.2. For every c > 1, there is $\Delta = \Delta(c)$, such that each of the following parameterized problems

k-Path, k-Vertex Cover, k-Tree, k-Steiner Tree, k-Internal Spanning Tree, k-Leaf Spanning Tree, k-Nonblocker, k-Path Contractibility, k-Cluster Editing, k-Set Splitting admits a Δ -polynomial formulation of complexity c^k .

4.1 Non-parameterized Problems

In the theorem below we prove that if a problem admits constant-degree polynomial formulations of complexity $2^{\gamma n}$ for every $\gamma > 0$, then SETH-hardness of the problem would imply a circuit lower bound.

Theorem 4.3. Let A be a computational problem. Assume that for every c > 1, there is $\Delta = \Delta(c)$ such that A admits a Δ -polynomial formulation of complexity c^n . If A is λ^n -SETH-hard for a constant $\lambda > 1$, then at least one of the following circuit lower bounds holds:

- E^{NP} requires series-parallel Boolean circuits of size $\omega(n)$;
- for every constant $\gamma > 1$, there exists an explicit family of constant-degree polynomials over \mathbb{Z} that requires arithmetic circuits of size $\Omega(n^{\gamma})$.

Proof. Let $\lambda > 1$ be the constant from the theorem statement, $\gamma > 1$ be an arbitrary constant, and $\sigma = \log(\lambda)/(6\gamma)$. For $n \in \mathbb{N}$, let I_n be the set of all instances of A of size n. Let \mathcal{P} be a Δ -polynomial formulation of A of complexity $2^{\sigma n}$, for constant $\Delta = \Delta(\sigma) > 0$. We assume that A is λ^n -SETH-hard: there is a function $\delta \colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that for every $k \in \mathbb{N}$, $(k\text{-SAT}, 2^n) \leq_{\delta} (P, \lambda^n)$.

We'll prove that at least one of the two circuit lower bounds holds. If $\mathcal{P} = (P_t)_{t\geq 1}$ does not have arithmetic circuits over \mathbb{Z} of size t^{γ} for infinitely many values of t, then we have an explicit family of constant-degree polynomials that requires arithmetic circuits of size $\Omega(t^{\gamma})$. Hence, in the following we assume that \mathcal{P} has arithmetic circuits over \mathbb{Z} of size ct^{γ} for all values of t for a constant c>0. Under this assumption, we design a non-deterministic algorithm solving k-TAUT in time $2^{(1-\varepsilon)n}$ for every k. This contradicts NSETH and, by Theorem 2.3, implies a super-linear lower bound on the size of series-parallel circuits computing E^{NP} .

Let $\delta_0 = \delta(1/2) \in (0,1)$ where δ is the function from the SETH-hardness reduction for A. Let $\alpha = \frac{1}{\gamma + 2\Delta + 8}$, $L = 2(1 - \delta_0)\alpha n/\log(\lambda)$, and $T = 2^{\sigma L}$. The meaning of these constants is the following. We will start with an instance of the k-TAUT problem on n variables, reduce it to $2^{(1-\alpha)n}$ instances of k-TAUT on αn variables each. Then we'll use the fine-grained reduction from k-SAT to the problem A on instances of size $\ell \leq L$. Finally, we'll use the polynomial formulation of A to reduce instances of size ℓ to polynomials with $t \leq T$ variables.

Let F be a k-DNF formula over n variables. In order to solve F, we branch on all but αn variables. This gives us $2^{(1-\alpha)n}$ k-DNF formulas. By solving k-SAT on the negations of all of these formulas, we solve k-TAUT on the original formula F.

We now apply the $(k\text{-}SAT, 2^n) \leq_{\delta} (A, \lambda^n)$ fine-grained reduction to (the negation of) each of the resulting formulas, which gives us a number of instances of A. Let ℓ be the largest size of these instances. From Definition 2.5, we know that for $\varepsilon = 1/2$ and $\delta_0 = \delta(1/2) > 0$, $\lambda^{\ell/2} = \lambda^{(1-\varepsilon)\ell} < 2^{(1-\delta_0)\alpha n}$, so each instance of A indeed has size less than $\ell < 2(1-\delta_0)\alpha n/\log(\lambda) = L$.

Since \mathcal{P} is a polynomial formulation of A of complexity $2^{\sigma n}$, there exist $s: \mathbb{N} \to \mathbb{N}, s(\ell) \leq 2^{\sigma \ell}$ and $\phi = (\phi_1, \phi_2, \dots)$ (computable in time $2^{\sigma L}$) such that for every ℓ and every $x \in I_{\ell}$,

- $P_{s(\ell)}(\phi_{\ell}(x)) \neq 0$ iff x is a yes instance of A;
- $|P_{s(\ell)}(\phi_{\ell}(x))| < 2^{s(\ell)}$.

Using \mathcal{P} , we will solve all instances of A in two stages: in the preprocessing stage (which takes place before all the reductions), we guess efficient arithmetic circuits for polynomials P_t for all $t \leq T$, in the solving stage, we solve all instances of A using the guessed circuits. Note that we'll

be using the polynomials to solve instances of A resulting from k-SAT instances on αn variables. Since L is the largest size of such an instance of A, we have that each such instance is mapped to a polynomial with at most $T = s(L) \leq 2^{\sigma L}$ variables. Therefore, finding efficient arithmetic circuits for polynomials P_t for all $t \leq T$ will be sufficient for solving the k-SAT instances of size αn .

Preprocessing. For every $t \leq T$, we find a prime p_t in the interval $2^{t+1} \leq p_t \leq 2^{t+2}$ in non-deterministic time $O(t^7)$ [AKS04, LP19].

Now for every $t \leq T$, we reduce all coefficients of the polynomial P_t modulo p_t to obtain a polynomial Q_t over \mathbb{Z}_{p_t} , and let $\mathcal{Q} = (Q_1, Q_2, \ldots)$. For every $t \leq T$, we now non-deterministically solve Gap-MACP $_{\mathcal{Q},\mu\Delta^2,p_t}(t,ct^{\gamma})$ using Lemma 3.5. Since we assume that \mathcal{P} has arithmetic circuits over \mathbb{Z} of size ct^{γ} , we have that \mathcal{Q} has arithmetic circuits over \mathbb{Z}_{p_t} of this size. Thus, we obtain arithmetic circuits C_t of size at most

$$c\mu\Delta^2 t^{\gamma}$$
 (1)

computing Q_t over \mathbb{Z}_p for all $t \leq T$. Since C_t computes Q_t correctly in \mathbb{Z}_p and $|P_{s(\ell)}(\phi_{\ell}(x))| < 2^{s(\ell)} \leq p_{s(\ell)}/2$ for all $x \in I_{\ell}$, we can use C_t to solve A for every instance size $\ell \leq L$. By Lemma 3.5, Gap-MACP_{$Q,\mu\Delta^2,p_t$} (t,ct^{γ}) can be solved in (non-deterministic) time

$$O\left(\Delta^2 \cdot ct^{\gamma} \cdot t^{2\Delta} \cdot \log^2(p_t)\right) = O\left(T^{\gamma+2\Delta+2}\right).$$

The total (non-deterministic) running time of the preprocessing stage is then bounded from above by the time needed to find T prime numbers, write down the corresponding explicit polynomials modulo p_t , and solve T instances of Gap-MACP:

$$O(T(T^7 + T^{\Delta+2} + T^{\gamma+2\Delta+2})) = O(T^{\gamma+2\Delta+8}) = O(2^{(1-\delta_0)n}),$$
 (2)

where the last equality holds due to $T = 2^{\sigma L}$, $L = 2(1 - \delta_0)\alpha n/\log(\lambda)$, $\sigma = \log(\lambda)/(6\gamma)$, and $\alpha = \frac{1}{\gamma + 2\Delta + 8}$.

Solving. In the solving stage, we solve all $2^{(1-\alpha)n}$ instances of k-SAT by reducing them to A and using efficient circuits found in the preprocessing stage. For an instance x of A of size ℓ , we first transform it into an input of the polynomial $y = \phi_{\ell}(x) \in \mathbb{Z}^{s(\ell)}$. Both $s(\ell)$ and $\phi_{\ell}(x)$ can be computed in time $O(2^{\sigma\ell})$. Then we feed it into the circuit $Q_{s(\ell)}$. First we note that we have the circuit $Q_{s(\ell)}$ after the preprocessing stage as $s(\ell) \leq s(L) \leq 2^{\sigma L} = T$ and we have circuits (Q_1, \ldots, Q_T) . The number of arithmetic operations in $\mathbb{Z}_{p_{s(\ell)}}$ required to evaluate the circuit is proportional to the circuit size, and each arithmetic operation takes time $\log^2(p_{s(\ell)}) = O(s(\ell)^2)$. From (1) with $t \leq s(\ell) \leq 2^{\sigma\ell}$, we have that we can solve an instance of A with ℓ inputs in time

$$O(2^{\sigma\ell}) + c\mu\Delta^2 \cdot s(\ell)^2 \cdot 2^{\sigma\gamma\ell} = O\left(2^{2\sigma\ell + \sigma\gamma\ell}\right) = O\left(2^{3\sigma\gamma\ell}\right) = O\left(\lambda^{\ell/2}\right),\,$$

where the last equality holds due to the choice of $\sigma = \log(\lambda)/(6\gamma)$. The fine-grained reduction from k-SAT to A implies that a $O\left(\lambda^{n/2}\right)$ -time algorithm for A gives us a $O\left(2^{n(1-\delta_0)}\right)$ -time algorithm for k-SAT. Thus, since we solve each ℓ -instance of A resulting from $2^{(1-\alpha)n}$ instances of k-SAT in time $O\left(\lambda^{\ell/2}\right)$, we solve the original n-variate instance F of k-TAUT in time

$$O\left(2^{(1-\alpha)n}\cdot(2^{\alpha n})^{1-\delta_0}\right) = O\left(2^{n(1-\alpha\delta_0)}\right). \tag{3}$$

The total running time of the preprocessing and solving stages (see (2) and (3)) is bounded from above by $O\left(2^{n(1-\delta_0)}\right) + O\left(2^{n(1-\alpha\delta_0)}\right) = O\left(2^{n(1-\alpha\delta_0)}\right)$, which refutes NSETH, and implies a super-linear lower bound for Boolean series-parallel circuits.

We now apply Theorem 4.3 to the non-parameterized problems from Section 2.6 to prove Theorem 1.1.

Theorem 1.1. If at least one of the following problems

k-SAT, MAX-k-SAT, Hamiltonian Path, Graph Coloring, Set Cover, Independent Set, Clique, Vertex Cover, 3d-Matching

is λ^n -SETH-hard for a constant $\lambda > 1$, then at least one of the following circuit lower bounds holds:

- E^{NP} requires series-parallel Boolean circuits of size $\omega(n)$;
- for every constant $\gamma > 1$, there exists an explicit family of constant-degree polynomials over \mathbb{Z} that requires arithmetic circuits of size $\Omega(n^{\gamma})$.

Proof. This follows immediately from Lemma 4.1 and Theorem 4.3.

4.2 Parameterized Problems

In the next theorem we show that if a parameterized problem admits constant-degree polynomial formulations of complexity $2^{\gamma k}$ for every $\gamma > 0$, then SETH-hardness of this problem would imply a circuit lower bound. The proof of Theorem 4.4 follows the high level strategy of the proof of Theorem 4.3, but takes into account the dependence on the parameter k of the parameterized problem under consideration and (arbitrary) polynomial dependence on the input length, we present the proof in Appendix A.1.

Theorem 4.4. Let A be a parameterized computational problem with a parameter k. Assume that for every c > 1, there is $\Delta = \Delta(c)$ such that A admits a Δ -polynomial formulation of complexity c^k . If A is λ^k -SETH-hard for a constant $\lambda > 1$, then at least one of the following circuit lower bounds holds:

- E^{NP} requires series-parallel Boolean circuits of size $\omega(n)$;
- for every constant $\gamma > 1$, there exists an explicit family of constant-degree polynomials over \mathbb{Z} that requires arithmetic circuits of size $\Omega(n^{\gamma})$.

We apply Theorem 4.4 to the parameterized problems from Section 2.6 to prove Theorem 1.2.

Theorem 1.2. If at least one of the following parameterized problems

k-Path, k-Vertex Cover, k-Tree, k-Steiner Tree, k-Internal Spanning Tree, k-Leaf Spanning Tree, k-Nonblocker, k-Path Contractibility, k-Cluster Editing, k-Set Splitting

is λ^k -SETH-hard for a constant $\lambda > 1$, then at least one of the following circuit lower bounds holds:

• E^{NP} requires series-parallel Boolean circuits of size $\omega(n)$;

• for every constant $\gamma > 1$, there exists an explicit family of constant-degree polynomials over \mathbb{Z} that requires arithmetic circuits of size $\Omega(n^{\gamma})$.

Proof. This follows immediately from Lemma 4.2 and Theorem 4.4.

While Theorem 1.2 conditionally rules out λ^k lower bounds for certain parameterized problems, we remark that the same machinery can be applied to conditionally rule out lower bounds of the form $n^{\lambda k}$ for constant $\lambda > 0$. We do not include rigorous proofs of such results in the paper as this would require us to generalize Theorem 4.4 to work with functions of k that may have different forms (exponential in k or in $k \log n$) at the expense of clarity of presentation. We note that the same techniques show that $n^{\gamma k}$ -SETH hardness of k-Clique or k-Independent Set for a constant $\gamma > 0$ would also imply one of the two circuit lower bounds. The polynomial formulations of parameterized k-Clique and k-Independent Set are identical to the polynomial formulations of the non-parameterized Independent Set problem presented in Section 6.1 with the only difference that the sizes of the sets S are now bounded by $2k/\theta$ instead of $2n/\theta$. This leads to $\binom{n}{\leq 2k/\theta} = n^{O(k)}$ variables in polynomial formulations and rules out $n^{\lambda k}$ lower bounds for the parameterized versions of k-Clique or k-Independent Set.

5 Deterministic Splitters over Alphabets of Linear Size

Our polynomial formulations of some of the problems (such as k-Path and k-Tree) will require deterministic constructions of certain splitters. This section is devoted to designing such splitters.

Definition 5.1. An (n, k, ℓ) -splitter H is a family of functions $f: [n] \to [\ell]$ such that for every set $S \subseteq [n]$ of size |S| = k, there exists a function $f \in H$ that splits S evenly:

$$\forall j \in [\ell], \ \lfloor k/\ell \rfloor \le f^{-1}(j) \le \lceil k/\ell \rceil$$
.

The set $[\ell]$ in this definition is called the alphabet. If $\ell \geq k$, an (n, k, ℓ) -splitter H is a family of functions from [n] to $[\ell]$ such that for every $S \subseteq [n]$, |S| = k, there exists an $f \in H$ which is injective on S. If $\ell = k$, then such a splitter is called a family of perfect-hash functions.

In this section, we present (n, k, ck)-splitters of size $\widetilde{O}(e^{\frac{k}{c}(1+o(1))})$ that can be computed in deterministic time $2^{9k}n^{O(1)}$.

It is easy to verify that a random set of $\approx e^{\frac{k}{c}}k\log n$ functions forms an (n,k,ck)-splitter with high probability. It's known [Fri84, Alo86] that a good linear code over the alphabet $[\ell]$ with relative distance $1 - \Theta(1/k^2)$ implies a splitter with related parameters. [Fri84, Alo86] use this observation to deterministically construct splitters of size $k^{O(1)}\log n$ for alphabets of size $\ell \geq k^2$. Although we can't use this splitter directly as we're working with alphabets of size $\ell = ck \ll k^2$, we'll use this primitive as one of the building blocks.

[NSS95, Theorem 3(iii)] gives a deterministic (n, k, ck)-splitter of size $\widetilde{O}(e^{k(1+o(1))})$. We follow the high-level approach of [NSS95] with certain low-level modifications to design a splitter of size $\widetilde{O}(e^{\frac{k}{c}(1+o(1))})$.

Our final construction of an (n, k, ck)-splitter will be a certain composition of splitters with various parameters. First, we give three auxiliary constructions of splitters with different parameters that will later be used in our main construction.

We say that an (n, k, ℓ) -splitter is explicit if the truth table of every function can be computed in deterministic time $(n\ell)^{O(1)}$. In particular, all functions of an explicit splitter H can be printed in time $|H|(n\ell)^{O(1)}$.

5.1 (n, k, k^2) -splitters

We present an efficient deterministic way to build (n, k, k^2) -splitters from [Fri84, Alo86] that will later effectively allow us to reduce the domain size from n to k^2 .

Proposition 5.2 ([Fri84, Alo86]). There is an explicit (n, k, k^2) -splitter $A(n, k, k^2)$ of size $O(k^6 \log k \log n)$.

Proof. There exist explicit linear codes [ABN⁺92] over the alphabet $[k^2]$ with at least n codewords, relative distance $\delta \geq 1 - 2/k^2$, and length $m = O(k^6 \log k \log n)$. Below we show that viewing such a code as a set of m functions from [n] to $[k^2]$ gives us the desired construction of an (n, k, k^2) -splitter.

Assume towards a contradiction that there exists a set T of k codewords such that for each of the m coordinates, a pair of codewords from T takes the same value at this coordinate. Then the sum of the $\binom{k}{2}$ pairwise distances between the codewords does not exceed $\binom{k}{2} \cdot m - m$. By averaging, there is a pair of codewords with distance at most

$$\frac{\binom{k}{2} \cdot m - m}{\binom{k}{2}} = m \left(1 - \frac{1}{\binom{k}{2}} \right) < m \left(1 - \frac{2}{k^2} \right) ,$$

which contradicts the assumption $\delta \geq 1 - 2/k^2$ on the relative distance of the code.

5.2 $(k^2, k, \log k)$ -splitters

Now we present explicit splitters of small size for the case of small alphabet $\ell = \log k$.

Proposition 5.3 ([NSS95, Lemma 4]). There is an explicit $(k^2, k, \log k)$ -splitter $B(k^2, k, \log k)$ of size $k^{2 \log k}$.

Proof. Let $\ell = \log k$. For each sequence $0 = i_0 < i_1 < \ldots < i_\ell = k^2$, define $f: [k^2] \to [\ell]$ by

$$f(x) = t \quad \text{iff} \quad i_{t-1} < x \le i_t \ .$$

This construction is explicit and has size $\binom{k^2}{\ell-1} \leq k^{2(\ell-1)} \leq k^{2\log k}$. In order to show that this is a (k^2,k,ℓ) -splitter, consider a set $S=\{j_1,\ldots,j_k\}\subseteq [n]$, where $j_1<\ldots< j_k$, and note that the function f defined by the set

$$i_1 = j_{k/\ell}, i_2 = j_{2k/\ell}, \dots, i_{\ell-1} = j_{(\ell-1)k/\ell},$$

splits the set S evenly.

5.3 $(k^2, k/\log k, ck/\log k)$ -splitters

Now we present a splitter with good parameters which is not explicit. We will later use it with small values of parameters so even though this splitter is not explicit, it will be possible to compute it in the allocated time. This primitive is based on the construction from [NSS95, Theorem 2(i)].

Lemma 5.4. There is an (n, k, ck)-splitter C(n, k, ck) of size $\widetilde{O}(e^{\frac{k}{c}(1+o(1))})$ that can be constructed deterministically in time $O((kn)^{3k})$.

Proof. Let T be a k-wise independent set of vectors of length n over the alphabet [ck]. There are explicit constructions of such sets of size $|T| \le n^k$ [AS08].

First we show that there exists $t \in T$ which, if viewed as a function $h: [n] \to [ck]$, splits at least an $e^{-k/c}$ fraction of k-sets of any family of k-sets $\mathcal{F} \subseteq {[n] \choose k}$. Indeed, a set $S \in \mathcal{F}$ is split by h if h is injective on S. The probability that h is injective on a fixed set of size k is

$$\left(1 - \frac{1}{ck}\right) \left(1 - \frac{2}{ck}\right) \cdots \left(1 - \frac{k-1}{ck}\right) \ge e^{-\frac{1}{ck} - \left(\frac{1}{ck}\right)^2 - \frac{2}{ck} - \left(\frac{2}{ck}\right)^2 - \dots - \frac{k-1}{ck} - \left(\frac{k-1}{ck}\right)^2} \ge e^{-k/c}.$$

Now, we iteratively greedily pick a vector from T splitting at least $e^{-k/c}$ -fraction of the remaining k-sets.

Size of the splitter. The size of the resulting splitter is at most smallest t satisfying

$$\binom{n}{k}(1 - e^{-k/c})^t \le 1.$$

That is, $t \leq e^{k/c} k \log n$.

Running time. The running time of each step of the greedy algorithm is at most $n \cdot \binom{n}{k} |T|$. And the total running time is at most

$$t \cdot n \cdot {n \choose k} \cdot |T| \le e^{k/c} \cdot n^k \cdot n^k \cdot (kn)^{O(1)} \le (kn)^{3k}$$
.

5.4 Main Construction

Equipped with the three auxiliary constructions above, we're in a position to present the main result of this section.

Theorem 5.5. For every $c \ge 1$, there is an (n, k, ck)-splitter of size $O(e^{\frac{k}{c}(1+o(1))}\log n)$ that can be constructed deterministically in time $2^{9k}n^{O(1)}$.

Proof. Let $A = A(n, k, k^2)$, $B = B(k^2, k, \log k)$, $C = C(k^2, k/\log k, ck/\log k)$ be the splitters from Propositions 5.2 and 5.3 and Lemma 5.4, respectively. Without loss of generality, we assume that k is a multiple of $\log k$. We define our (n, k, ck)-splitter H as follows. For every function $a \in A$, every function $b \in B$, and every $\log k$ -tuple of functions $(h_1, \ldots, h_{\log k})$ from C, H contains the function $f: [n] \to [ck]$, where

$$f(x) = \frac{ck}{\log k} \cdot b(a(x)) + h_{b(a(x))}(a(x)). \tag{4}$$

Correctness. Let $S \subseteq [n]$ be a set of size |S| = k. We will show that there exist functions $a \in A$, $b \in B$, and $(h_1, \ldots, h_{\log k})$ in $C^{\log k}$ such that for f defined in (4) and every pair of distinct $s_1, s_2 \in S$, $f(s_1) \neq f(s_2)$. Equivalently, for $s \in S$, if y = b(a(s)) and $S_y = \{s \in S : b(a(s)) = y\}$, then h_y is injective on $a(S_y)$.

Since A and B are splitters, there exist $a \in A$ and $b \in B$ such that for every $y \in [\log k]$, $|S_y| \le k/\log k$. Now since C is a splitter and $|a(S_y)| \le |S_y| \le k/\log k$, we have that for every $y \in [\log k]$, there exists $h_y \in C$ such that h_y is injective on $a(S_y)$. Therefore, the function f defined with the selected $a, b, h_1, \ldots, h_{\log k}$ satisfies the requirement that $f(s_1) \ne f(s_2)$ for all distinct $s_1, s_2 \in S$.

Size of the splitter. By the definition of H,

$$|H| = |A| \cdot |B| \cdot |C|^{\log k} = O(k^6 \log k \log n) \cdot O(k^{2 \log k}) \cdot \left(O\left(e^{\frac{k}{c \log k}(1 + o(1))}\right)\right)^{\log k} = O\left(e^{\frac{k}{c}(1 + o(1))} \log n\right) \ .$$

Running time. The splitters from Propositions 5.2 and 5.3 are explicit, and the splitter from Lemma 5.4 takes time $\left(k^2 \cdot \frac{k}{\log k}\right)^{\frac{3k}{\log k}} = O(2^{9k})$.

6 Polynomial Formulations

In this section, we prove Lemmas 4.1 and 4.2: we give polynomial formulations of all problems from Section 2.6.

6.1 Non-parameterized Problems

Lemma 4.1. For every c > 1, there is $\Delta = \Delta(c)$, such that each of the following problems

k-SAT, MAX-k-SAT, Hamiltonian Path, Graph Coloring, Set Cover, Independent Set, Clique, Vertex Cover, 3d-Matching

admits a Δ -polynomial formulation of complexity c^n .

Proof. Let A be one of the problems from the list above, and for every $n \in \mathbb{N}$, let I_n be the set of instances of A of size n. We construct a family of mappings $\phi = (\phi_1, \phi_2, \dots)$, where $\phi_n : I_n \to \{0,1\}^{s(n)}$ and a family of polynomials $\mathcal{P} = (P_1, P_2, \dots)$, following the same five-step pattern.

Idea. We provide a high-level idea of encoding a problem as a polynomial. Start by fixing a parameter $\theta = \theta(c)$ that will be chosen as a large enough constant. In the analysis, we write n/θ instead of $\lceil n/\theta \rceil$: this affects the bounds negligibly and at the same time simplifies the bounds. Then, a solution of size n that we are looking for can be broken into "blocks" of size n/θ ; for each potential block, we introduce a 0/1-variable; then, for each candidate solution, we introduce a monomial that is non-zero if it is indeed a solution.

Variables. We introduce a set X of s(n) variables. They are used to specify the function ϕ_n that maps an instance $I \in I_n$ to a vector in $\mathbb{Z}^{s(n)}$. To do this, we specify a 0/1-value that $\phi_n(I)$ assigns to every variable $x \in X$.

Complexity. We bound the number of variables s(n) of the constructed polynomial $P_{s(n)}$ as well as the time needed to compute the mapping ϕ_n by

$$\left(2^{n/\theta}\cdot \binom{n}{n/\theta}\right)^{O(1)}$$
.

In all the cases, it will be straightforward to compute s(n) in the allocated time.

Polynomial. We specify the polynomial $P_{s(n)}(X)$ as a sum of $2^{O(n)}$ monomials (where the hidden constant in O(n) depends on $\theta = \theta(c)$ only), each having coefficient 1. It is usually straightforward from the definition of the polynomial that I is a yesinstance of A iff $P_{s(n)}(\phi_n(I)) > 0$.

Degree. We show that the degree Δ of \mathcal{P} depends on θ only.

Below, we show that the five steps above ensure that \mathcal{P} is indeed a polynomial formulation of A.

• By choosing a large enough $\theta = \theta(c)$, we ensure that, for all large enough n,

$$|X| = s(n) = \left(2^{n/\theta} \cdot \binom{n}{n/\theta}\right)^{O(1)} < c^n.$$

- Since $P_{s(n)}(X)$ is a sum of $2^{O(n)}$ monomials, computing the coefficients of all monomials in $P_{s(n)}$ takes time $2^{O(n)}$. Since $|X| = c^n$, $2^{O(n)} = |X|^{O(1)}$. Since the degree of \mathcal{P} is a Δ -explicit family of polynomials.
- Recall that ϕ_n maps an instance of the problem to a vector from $\{0,1\}^{s(n)}$, and that all the coefficients of the polynomials $P_{s(n)}$ are ones. Thus, for every $I \in I_n$, $|P_{s(n)}(\phi_n(I))|$ is at most the number of monomials in $P_{s(n)}$, i.e., $|X|^{O(1)}$, and hence at most $2^{|X|}$. Since $\phi_n(I)$ can be computed in time c^n , we conclude that \mathcal{P} is indeed a polynomial formulation of A.

Hamiltonian Path. Given a directed graph G(V, E) with n nodes, check whether it contains a Hamiltonian path.

Idea. One can break a Hamiltonian path π into θ node-disjoint paths $\pi_1, \ldots, \pi_{\theta}$ of length n/θ each. We say that $\pi_1, \ldots, \pi_{\theta}$ is a valid partition iff π_i 's are simple paths of length n/θ sharing no nodes and, for every i, there is an edge joining the last node of π_i with the first node of π_{i+1} .

Variables. Introduce $s(n) = O\left(n^2 \binom{n}{n/\theta}\right)$ variables:

$$X = \left\{ x_{S,u,v} \colon S \subseteq V, |S| = n/\theta, u \in S, v \in V \setminus S \right\}.$$

The mapping $\phi_n(G)$ assigns the following 0/1-value to a variable $x_{S,u,v}$:

[there is a Hamiltonian path in G[S] that starts at u and ends at a node adjacent to v]

(here and below, $[\cdot]$ is the Iverson bracket: for a predicate Y, [Y] = 1 if Y is true and [Y] = 0 otherwise).

Complexity. The mapping $\phi_n(G)$ can be computed in time $O^*(\binom{n}{n/\theta})2^{n/\theta}$ because Hamiltonian Path on a graph with n nodes can be solved in time $O^*(2^n)$.

Polynomial. For every partition $V = S_1 \sqcup \cdots \sqcup S_\theta$ into disjoint subsets of size n/θ and every θ nodes $v_1, \ldots, v_{\theta+1}$, add to $P_{s(n)}$ a monomial

$$x_{S_1,v_1,v_2} \cdot x_{S_2,v_2,v_3} \cdot \cdot \cdot x_{S_{\theta},v_{\theta},v_{\theta+1}}$$

The number of monomials added to $P_{s(n)}$ is at most $\theta^n n^{\theta} = 2^{O(n)}$.

Degree. The degree of \mathcal{P} is θ .

3d-Matching. Given a 3-uniform 3-partite hypergraph $G(V_1 \sqcup V_2 \sqcup V_3, E)$ with parts of size n (that is, $|V_1| = |V_2| = |V_3| = n$ and $E \subseteq V_1 \times V_2 \times V_3$) and an integer t, check whether G contains a matching of size at least t.

Idea. One can break a matching M of size t into θ matchings M_1, \ldots, M_{θ} of size at most n/θ . Then, M_1, \ldots, M_{θ} is a valid partition iff M_i 's are node-disjoint matchings.

Variables. Introduce $s(n) = \binom{n}{\leq n/\theta}^3 = O^* \left(\binom{n}{n/\theta}^3\right)$ variables:

$$X = \{x_{A,B,C} : A \subseteq V_1, B \subseteq V_2, C \subseteq V_3, |A| = |B| = |C| \le n/\theta\}.$$

The function $\phi_n(G)$ assigns the following value to $x_{A,B,C}$:

[the induced subgraph $G[A \sqcup B \sqcup C]$ contains a perfect matching].

Complexity. This can be computed in time $O^*\left(8^{n/\theta}\binom{n}{n/\theta}\right)^3$ since 3d-Matching is solvable in time $O^*(8^n)$ in 3-partite graphs with parts of size n.⁴

Polynomial. The polynomial $P_{s(n)}$ is defined as follows. For every $A_1, \ldots, A_{\theta} \subseteq V_1, B_1, \ldots, B_{\theta} \subseteq V_2, C_1, \ldots, C_{\theta} \subseteq V_3$, such that all A_i 's, B_i 's, and C_i 's are pairwise disjoint, have size at most n/θ , and that

$$\left| \bigcup_{i \in [\theta]} A_i \right| = t \,,$$

add to $P_{s(n)}$ a monomial

$$\prod_{j\in[\theta]}x_{A_j,B_j,C_j}.$$

The number of monomials added to $P_{s(n)}$ is at most $\binom{n}{n/\theta}^{3\theta} = 2^{O(n)}$.

Degree. The degree of \mathcal{P} is θ .

⁴This is done by a straightforward dynamic programming algorithm: for $A \subseteq V_1$, $B \subseteq V_2$, $C \subseteq V_3$, let M(A, B, C) be the maximum size of a matching in $G[A \cup B \cup C]$; then, $M(A, B, C) = \max_{\{a,b,c\} \in E} M(A \setminus a, B \setminus b, C \setminus c) + 1$.

Independent Set. Given a graph G(V, E) with n nodes and an integer t, check whether G contains an independent set of size at least t.

Idea. An independent set I of size t can be partitioned into θ sets I_1, \ldots, I_{θ} of size at most n/θ . Then, I_1, \ldots, I_{θ} is a valid partition iff their total size is at least t and for all $i \neq j$, $I_i \cup I_j$ is an independent set.

Variables. Introduce $s(n) = \binom{n}{\langle 2n/\theta \rangle} = O^*(\binom{n}{2n/\theta})$ variables:

$$X = \{x_S \colon S \subseteq V \text{ and } |S| \le 2n/\theta\}.$$

The mapping $\phi_n(G)$ assigns to x_S the value

[S is an independent set of G].

Complexity. The mapping $\phi_n(G)$ can be computed in time $O^*(2^{2n/\theta}\binom{n}{2n/\theta})$, since Independent Set can be solved in time $O^*(2^n)$.

Polynomial. For every $S_1, \ldots, S_{\theta} \in \binom{V}{\leq n/\theta}$ such that $S_i \cap S_j = \emptyset$, for all $i \neq j$, and $|\bigcup_{i \in [\theta]} S_i| = t$, add to $P_{s(n)}$ a monomial

$$\prod_{1 \le i < j \le \theta} x_{S_i \cup S_j} .$$

The number of monomials added to P is at most $\theta^n = 2^{O(n)}$.

Degree. The degree of \mathcal{P} is $O(\theta^2)$.

Vertex Cover and Clique. These two problems are close relatives of *Independent Set*: the complement of an independent set in a graph is a vertex cover of this graph; a clique in a graph is an independent set in the complement of the graph. Thus, for *Vertex Cover* and *Clique* one can use the polynomial formulation of *Independent Set*.

MAX-k**-SAT.** Given a k-CNF formula $F = C_1 \wedge \cdots \wedge C_m$ over n variables and an integer t, check whether it is possible to satisfy at least t clauses of F.

Idea. An assignment $\mu \in \{0,1\}^n$ satisfying at least t clauses can be partitioned into θ subassignments $\mu_1, \ldots, \mu_{\theta} \in \{0,1\}^{n/\theta}$. Then, for each clause C, one can assign at most k subassignments that are "responsible" for C: these are the subassignments that contain the k variables from C. Then, $\mu_1, \ldots, \mu_{\theta}$ is a valid partition if the total number of clauses satisfied by their k-tuples of subassignments is at least t.

Variables. Partition the set of variables of F into θ blocks V_1, \ldots, V_{θ} of size n/θ . For each clause C, assign k blocks such that all variables of C belong to these blocks: formally, let $b(C) \subseteq [\theta]$, |b(C)| = k, and the set of variables of C_i is a subset of $\bigcup_{i \in b(C)} V_i$.

Introduce $s(n) = \binom{n/\theta}{k} 2^{nk/\theta} t$ variables:

$$X = \{x_{B,\tau,r} \colon B \subseteq [\theta], |B| = k, \tau \in \{0,1\}^{nk/\theta}, 0 \le r \le t\}.$$

For $B \subseteq [\theta]$, |B| = k, by c(B) define the set of clauses C of F such that b(C) = B. The mapping $\phi_n(F)$ assigns the following value to $x_{B,\tau,r}$:

 $[\tau \text{ satisfies at least } r \text{ clauses from } c(B)].$

Polynomial. Let \mathcal{F} be a class of functions $f : 2^{[\theta]} \to \mathbb{Z}_{\geq 0}$ such that $\sum_{B \subseteq [\theta], |B| = k} f(B) = t$. For $\mu \in \{0, 1\}^n$ and $B \subseteq [\theta]$, let μ_B be a projection on coordinates $\bigcup_{i \in B} V_i$. For every $f \in \mathcal{F}$ and $\mu \in \{0, 1\}^n$, add to $P_{s(n)}$ a monomial

$$\prod_{B\subseteq [\theta], |B|=k} x_{B,\mu_B,f(B)}.$$

Clearly, $|\mathcal{F}| \leq t^{2^{\theta}} \leq n^{O(k2^{\theta})}$ and $|\{B : B \subseteq [\theta], |B| = k\}| \leq 2^{\theta}$. Hence, the number of monomials added to P is at most $O^*(2^n) = 2^{O(n)}$.

Degree. The degree of P is $\binom{\theta}{k} \leq 2^{\theta}$.

k-SAT. k-SAT is a special case of MAX-k-SAT.

Graph Coloring. Given a graph G(V, E) with n nodes and an integer t, check whether G can be colored properly using at most t colors.

Idea. Partition V into θ blocks V_1, \ldots, V_{θ} of size n/θ . We would like to construct a t-coloring of V from colorings of the blocks. However, a t-coloring may contain a color whose color class is large (much larger than n/θ). For this reason, the polynomial formulation below is a bit trickier than the previous ones: we have to treat color classes differently depending on their size.

Variables. Introduce $s(n) = O(n^2 \binom{n}{2n/\theta})$ variables:

$$X = \left\{ x_{S,r} \colon S \subseteq V, |S| \le 2n/\theta, 0 \le r \le t \right\}.$$

The mapping $\phi_n(G)$ assigns to $x_{S,r}$ the value

$$\left[\chi(G[S]) \le r\right].$$

As the chromatic number of a *n*-node graph can be found in time $O^*(2^n)$ [BHK09], the mapping $\phi_n(G)$ can be computed in time

$$O^*\left(2^{2n/\theta}\binom{n}{2n/\theta}\right)$$
.

Polynomial. The polynomial $P_{s(n)}$ is defined as follows:

$$P_{s(n)}(X) = \sum_{\substack{V = S_1 \sqcup \cdots \sqcup S_\theta \\ |S_i| \leq 2n/\theta \text{ for } i \in [\theta]}} \sum_{\substack{t_1 + \cdots + t_p = t \\ i \in [p]}} \prod_{i \in [p]} x_{S_i, t_i}.$$

We claim that $P_{s(n)}(\phi_n(X)) > 0$ iff G can be properly colored using t colors such that every color induces an independent set of size at most n/θ . Indeed, if there is such a coloring, one can greedily pack color classes into groups of size at most $2n/\theta$ and obtain the required partition $V = S_1 \sqcup \cdots \sqcup S_\theta$.

Thus, it remains to consider the case when there exists a t-coloring where at least one of the color classes has size more than n/θ . Since each color class induces an independent set in the graph, we are going to reuse the ideas from polynomial formulation of Independent Set. Namely, assume that T_1, \ldots, T_l are all large color classes in a t-coloring: for every $i \in [l]$, $T_i \subseteq V$ is an independent set of size more than n/θ .

Introduce the following additional variables:

$$Y = \{y_S \colon S \subseteq V, |S| \le 2n/\theta\}.$$

The mapping $\phi_n(G)$ assigns the following values to y_S :

[S is an independent set in G].

For every $L \subseteq V$ of size $l > n/\theta$, fix its partition (say, the lexicographically first one) $L_1 \sqcup \cdots \sqcup L_{\lceil l\theta/n \rceil}$ into subsets of size n/θ : all sets are disjoint and all of them have size n/θ except for possibly the last one. The following monomial expresses the fact that L is an independent set in G:

$$M(L) = \prod_{1 \le i < j \le \lceil l\theta/n \rceil} y_{L_i \cup L_j}.$$

The final polynomial looks as follows:

$$Q_{s(n)}(X,Y) = \sum_{l=0}^{\theta} \sum_{\substack{T_1, \dots, T_l \subseteq V \\ T_i \cap T_j = \emptyset \text{ for all } i \neq j \\ |T_i| > n/\theta \text{ for all } i}} \sum_{\substack{S_1 \sqcup \dots \sqcup S_\theta = V \setminus \bigcup_{i \in [l]} T_i \\ t_1 + \dots + t_n = t-l}} \prod_{i \in [p]} M(T_i) \prod_{i \in [\theta]} x_{S_i, t_i}.$$

The number of monomials added to Q is at most $O^*(n^{\theta}\theta^n) = 2^{O(n)}$.

Degree. The degree of $Q_{s(n)}$ is at most $2\theta^2 + \theta$.

Set Cover. Given a set family $\mathcal{F} = \{F_1, \dots, F_m\} \subseteq 2^{[n]}, m = n^{O(1)}$ and an integer t, check whether one can cover [n] with at most t sets from \mathcal{F} .

Idea. Partition the universe [n] into θ blocks of size n/θ . Each of these blocks is either covered by at most t sets or is covered by a single large set (of size at least n/θ) that also possibly intersects other blocks.

Variables. Introduce $s(n) = O(n^{O(1)} \binom{n}{2n/\theta})$ variables:

$$X = \{x_{S,r} : S \subseteq [n], |S| \le 2n/\theta, 0 \le r \le t\},$$

$$Y = \{y_{S,i} : S \subseteq [n], |S| \le 2n/\theta, 1 \le i \le m\}.$$

The mapping $\phi_{s(n)}(\mathcal{F})$ assigns the following values to $x_{S,r}$ and $y_{S,i}$:

[S can be covered by r sets from
$$\mathcal{F}$$
],
[S $\subseteq F_i$],

Complexity. As Set Cover problem can be solved in time $O^*(2^n)$ [BHK09], the mapping $\phi_{s(n)}(\mathcal{F})$ can be computed in time

$$O^*\left(2^{\frac{2n}{\theta}}\binom{n}{\frac{2n}{\theta}}\right)$$
.

Polynomial. For every $L \subseteq V$ of size $l > n/\theta$, fix its partition (say, the lexicographically first one) $L_1 \sqcup \ldots \sqcup L_{\lceil \theta l/n \rceil}$ into subsets of size n/θ : all sets are disjoint and all of them have size n/θ except for possibly the last one. The following monomial expresses the fact that $L \subseteq F_q$:

$$M(L,q) = \prod_{i=1}^{\lceil \theta l/n \rceil} y_{L_i,q}.$$

Finally, the polynomial $Q_{s(n)}(X,Y)$ is defined as follows:

$$\sum_{l=0}^{\theta} \sum_{\substack{q_1, \dots, q_l \in [m] \\ \forall i \neq j: q_i \neq q_j}} \sum_{\substack{T_1, \dots, T_l \subseteq V \\ T_i \cap T_j = \emptyset \text{ for all } i \neq j}} \sum_{\substack{S_1 \sqcup \dots \sqcup S_\theta = V \setminus \bigcup_{i \in [l]} T_i \ i \in [l] \\ t_1 + \dots + t_p = t - l}} \prod_{i \in [l]} M(T_i, q_i) \prod_{i \in [\theta]} x_{S_i, t_i}.$$

Degree. The degree of this polynomial is at most $(\theta+1)\theta$. The number of monomials added to $Q_{s(n)}$ is at most $O^*(n^{O(\theta)}\theta^n) = 2^{O(n)}$.

6.2 Parameterized Problems

6.2.1 Technical Lemmas

For parameterized polynomial formulations, we utilize the following technical lemma. We provide its proof in Appendix A.2.

Lemma 6.1. Let T(V, E) be a tree, $M \subseteq V$ be a set of k nodes, and $\theta > 1$ be an integer. Then E can be partitioned into $m \leq \theta$ blocks $E = E_1 \sqcup \cdots \sqcup E_m$ such that, for each $i \in [m]$, E_i induces a subtree T_i of T with at most $\frac{2k}{\theta - 1} + 2$ nodes from M.

The following definition resembles a block structure of a graph: for a family of sets X_1, \ldots, X_m we introduce m nodes and connect the i-th of these nodes with all elements of X_i that belong to at least one other X_i .

Definition 6.2. Given m sets X_1, \ldots, X_m , we introduce a set $S = \{s_1, \ldots, s_m\}$ such that S does not intersect any X_i , and a set $C = \bigcup_{i \neq j} (X_i \cap X_j)$. By a subset graph $B(X_1, \ldots, X_m)$ we denote the following graph G(V, E): $V = S \sqcup C$, $E = \{\{s_i, c\}: c \in X_i \cap C\}$. Each $v \in S$ is called a set node, and each $v \in C$ is called a connector node.

Remark 6.3. Let T_1, \ldots, T_m be the trees resulting from applying Lemma 6.1 to a tree T. Then, $B = B(V(T_1), \ldots, V(T_m))$ is a tree containing m set nodes and at most m-1 connector nodes.

6.2.2 Polynomial Formulations

Lemma 4.2. For every c > 1, there is $\Delta = \Delta(c)$, such that each of the following parameterized problems

k-Path, k-Vertex Cover, k-Tree, k-Steiner Tree, k-Internal Spanning Tree, k-Leaf Spanning Tree, k-Nonblocker, k-Path Contractibility, k-Cluster Editing, k-Set Splitting

admits a Δ -polynomial formulation of complexity c^k .

Proof. We will design polynomial formulations $P_{s(x,k)}$ for the above problems where s(x,k) = s(n,k) will be a function of n = |x| and k. Naturally, we would like to have different polynomials P for different values of (n,k), alas, s is not necessarily injective. One way to overcome this issue is to consider a two-dimensional sequence of polynomials $P_{s(n,k),k}$ as we'll always have that $s(\cdot,k)$ is injective for every k. But this approach would cause technical issues in the proof of arithmetic lower bounds in Theorem 4.4. Instead, we still consider a sequence (P_1, P_2, \ldots) of polynomials, but slightly modify the function s. Given a function s(n,k), we define the following Cantor pairing function s(n,k) and s:

$$s'(n,k) = (s(n,k) + k)(s(n,k) + k + 1)/2 + k.$$

We note that $s'(\cdot, \cdot)$ is injective because $s(\cdot, k)$ is injective for every k. In all polynomial formulations below, we can always switch from s(x,k) to s'(x,k) to resolve the aforementioned issue. Indeed, since $s'(x,k) \geq s(x,k)$, we can just add s'(x,k) - s(x,k) dummy variables to the polynomial. Also, since $s'(x,k) \leq (s(x,k)+k)^2$, it suffices to replace the bound $s(x,k) \leq T(k)|x|^{\Delta}$ by the bound $s'(x,k) \leq (s(x,k)+k)^2 \leq T'(k)|x|^{\Delta'}$ for $T'(x) = T(k)^2 \cdot k^2$ and $\Delta' = \Delta$. Since we're proving polynomial formulations for all $T = c^k, c > 1$, the change in T' doesn't affect the statement of the lemma.

In all polynomial formulations below we follow the same five-step pattern as in Lemma 4.1 with the following three differences.

Kernel. For most of the considered problems, we start by applying a kernel. Recall that a kernel replaces, in polynomial time, an instance (x, k) of a parameterized problem A with an equivalent instance (x', k') of A such that $|x'|, k' \leq g(k)$, for some computable function g. To simplify the presentation of polynomial formulations, we identify (x, k) with (x', k'). This allows us to assume from the beginning that $n = |x| \leq g(k)$.

Variables. The value of s(x,k) will be a constant degree of a product of $2^{O(k/\theta)}$ and $\binom{O(k)}{O(k/\theta)}$ as well as $n^{O(1)}$. By choosing a large enough $\theta = \theta(c)$, we ensure that $s(x,k) \leq c^k n^{O(1)}$.

Polynomial. We ensure that the polynomial $P_{s(x,k)}$ is a sum of at most $n^{O(1)}2^{O(k)}$ monomials.

k-Vertex Cover. Given a graph G(V, E), check whether G contains a vertex cover of size at most k.

Kernel. As there exists a kernel of size 2k for k-Vertex Cover [CKJ01], we assume that $|V| = n \le 2k$.

Idea. Partition V into θ blocks of size n/θ : $V = V_1 \sqcup \cdots \sqcup V_\theta$. This induces a partition of any $S \subseteq V$ into θ blocks of size at most n/θ : $S = S_1 \sqcup \cdots \sqcup S_\theta$ where $S_i = S \cap V_i$. Then, S is a vertex cover of G iff, for all $1 \leq i < j \leq \theta$, $S_i \sqcup S_j$ is a vertex cover of $G[V_i \sqcup V_j]$.

Variables. Introduce $s(G, k) = O(n^2 \binom{n}{\langle n/\theta \rangle^2}) = O^*(\binom{2k}{2k/\theta})$ variables:

$$X = \{x_{i,j,A,B} \colon i, j \in [\theta], A \subseteq V_i, B \subseteq V_j\}.$$

The mapping $\phi(G, k)$ assigns the following value to a variable $x_{i,j,A,B}$:

$$[A \sqcup B \text{ is a vertex cover of } G[V_i \sqcup V_i]].$$

Complexity. The mapping $\phi(G,k)$ can be computed in time $O^*(\binom{2k}{2k/\theta})$.

Polynomial. For every $S \subseteq V$ of size at most k, add the following monomial to $P_{s(G,k)}$:

$$\prod_{1 \le i < j \le \theta} x_{i,j,S_i,S_j} .$$

The number of monomials added to $P_{s(G,k)}$ is $\binom{n}{\langle k} = O^*(\binom{2k}{k})$.

Degree. The degree of \mathcal{P} is at most θ^2 .

k-Steiner Tree. Given a graph G(V, E) with (integer non-negative) edge weights and a subset $S \subseteq V$ of its nodes of size k (called terminals), and an integer $0 \le t \le |V|^{O(1)}$, check whether there is a tree in G of weight at most t containing all nodes from S.

Idea. Assume that $S = \{1, \ldots, k\}$: relabel nodes if needed. Consider a Steiner tree T that we are looking for. Using Lemma 6.1 for T and M = S one can find subtrees T_1, \ldots, T_m of T for some $m \leq \theta$ such that each T_i contains at most $\frac{2k}{\theta-1} + 2 \leq 3k/\theta$ terminals. Let V_1, \ldots, V_m be the corresponding sets of nodes, that is, $V_i = V(T_i)$, let $S_i = S \cap V_i$ for every $1 \leq i \leq m$, let ℓ_i be the weight of T_i , and let $B = B(V_1, \ldots, V_m)$. By Remark 6.3, B is a tree and it contains m subset nodes and at most m-1 connector nodes. Observe that B can also be obtained as $B(A_1, \ldots, A_m)$, where $A_i = V_i \cap \bigcup_{j \neq i} V_j$. It is significant since each $|A_i|$ is bounded by $\theta - 1$

in contrast to $|V_i|$ which is bounded by n. Note also that $\sum \ell_i \leq L$ and for every i, a subtree T_i has weight ℓ_i and connects nodes from $S_i \cup A_i$.

To construct the polynomial, we go over all possible ℓ_i , $\{S_i\}$ and $\{A_i\}$ such that $\sum \ell_i \leq L$, $\bigcup S_i = S$ and $B(A_1, \ldots, A_m)$ is connected, and check whether for every i we can connect nodes from $S_i \cup A_i$ with a tree of weight at most ℓ_i . Observe that those trees can intersect and their union can give us a connected subgraph that is not a tree, but we still can obtain a proper Steiner tree just by taking a spanning tree of that subgraph.

Variables. Introduce $s(G, k) = \mathcal{O}(\binom{k}{\leq 3k/\theta + \theta}) \cdot n^{\theta} \cdot n^{O(1)} + n^{O(1)} = O^*(\binom{k}{4k/\theta})$ variables:

$$X = \{x_{S',A,\ell} \colon S' \subseteq S, |S'| \le 3k/\theta, A \subseteq V, |A| \le \theta - 1, \ell \le n^{O(1)}\};$$
$$Y = \{y_{L'} \colon L' \le n^{O(1)}\}.$$

The mapping $\phi(G, k)$ assigns the following values to variables:

 $x_{S',A,\ell} \mapsto [\text{there exists a Steiner tree in } G \text{ for the set of terminals } S' \cup A$ of the weight at most $\ell];$ $y_{L'} \mapsto [L' \leq L].$

Complexity. The mapping $\phi(G, k)$ can be computed in time $O^*(\binom{k}{4k/\theta}) \cdot 2^{4k/\theta}$, since k-Steiner tree can be solved in time $O^*((2+\delta)^k)$ for any $\delta > 0$ [FKM⁺07].

Polynomial. The polynomial looks as follows:

$$P_{s(G,k)}(X,Y) = \sum_{\substack{m \leq \theta, \\ \ell_1, \dots, \ell_m, \\ \sum \ell_i \leq n^{O(1)}}} \sum_{\substack{S_1 \cup \dots \cup S_m = S, A_1, \dots, A_m \subseteq V, \\ |S_i| \leq 3k/\theta}} \sum_{\substack{l \cup A_i \leq \theta - 1, \\ B(A_1, \dots, A_m) \\ \text{is connected}}} y_{\sum \ell_i} \prod_{i=1}^m x_{S_i, A_i, \ell_i}.$$

The number of monomials in $P_{s(G,k)}$ is at most $\theta \cdot n^{O(1)} \cdot \binom{k}{\leq 3k/\theta}^{\theta} \cdot \binom{n}{\theta}^{\theta} = 2^{O(k)} n^{O(1)}$.

Degree. The degree of \mathcal{P} is $\theta + 1$.

k-Internal Spanning Tree. Given a graph G, check whether there is a spanning tree of G with at least k internal nodes.

Kernel. As there exists a kernel for k-Internal Spanning Tree of size 3k [FGST13] we assume that $|V| = n \le 3k$.

Idea. Let T be a tree we are looking for. Using Lemma 6.1 for T and M = V(T), we obtain $m \le \theta$ subtrees T_i of size at most $\frac{2n}{\theta-1}+2 \le 3n/\theta$. For each $i \in [m]$, let $S_i = V(T_i)$ and k_i be the number of internal nodes of T that belong to S_i . By Remark 6.3, $B = B(S_1, \ldots, S_m)$ is a tree. Let S be the set of set nodes and C be the set of connector nodes of S. When summing up all S_i , we count each internal node S_i of S_i that belongs to S_i exactly S_i times. Then, we count S_i deg S_i deg S_i exactly deg S_i to S_i exactly deg $S_$

To construct the polynomial, we go over all possible $S_1, \ldots, S_m \subseteq V$ such that $\bigcup S_i = V$ and $B(S_1, \ldots, S_m)$ is a tree and over all possible k_1, \ldots, k_m such that $\sum k_i \geq k + (m-1)$. For fixed $\{S_i\}$ and $\{k_i\}$, it suffices to check that for every i there exists a spanning tree of $G[S_i]$ with at least k_i internal nodes, where we also should consider $S_i \cap \bigcup_{j \neq i} S_j$ as internal nodes (to

do that, we add a leaf to each node from $S_i \cap \bigcup_{j \neq i} S_j$).

Variables. Introduce $s(G, k) = \binom{n}{\leq 3n/\theta} \cdot \binom{n}{\leq \theta} \cdot 3n/\theta = O^*(\binom{3k}{9k/\theta})$ variables:

$$X = \{x_{S,A,k'} \colon A \subseteq S \subseteq V, |S| \le 3n/\theta, |A| \le \theta - 1, k' \le |S|\}.$$

Let $G_{S,A}$ be a graph obtained from G[S] by adding a leaf to each node $v \in A$. The mapping $\phi(G,k)$ assigns the following value to a variable $x_{S,A,k'}$:

[there exists a spanning tree of $G_{S,A}$ that contains at least k' internal nodes].

Complexity. The mapping $\phi(G, k)$ can be computed in time $O^*(\binom{3k}{9k/\theta}) \cdot 8^{3n/\theta}) = O^*(\binom{3k}{9k/\theta}) \cdot 8^{9k/\theta}$, since k-Internal Spanning Tree can be solved in time $O^*(8^k)$ [FGST13].

Polynomial. The polynomial looks as follows:

$$P_{s(G,k)}(X) = \sum_{m \in [\theta]} \sum_{\substack{S_1, \dots, S_m \subseteq V, \\ 1 < |S_i| \le 3n/\theta, \\ \bigcup S_i = V, \\ B(S_1, \dots, S_m) \text{ is a tree}}} \sum_{\substack{k_1, \dots, k_m, \\ k_i \le |S_i|, \\ \sum k_i \ge k + (m-1)}} \prod_{i=1}^m x_{S_i, \bigcup_{j \ne i} (S_i \cap S_j), k_i}$$

The number of monomials in $P_{s(G,k)}$ is at most $\theta \cdot \binom{n}{3n/\theta}^{\theta} \cdot (3n/\theta)^{\theta} = \binom{3k}{9k/\theta}^{\theta} k^{O(1)} = 2^{O(k)}$. Degree. The degree of \mathcal{P} is θ .

k-Leaf Spanning Tree. Given a graph G, check whether there is a spanning tree of G with at least k leaves.

Kernel. As there exists a kernel for k-Leaf Spanning Tree of size 5.75k [FMRU00], we assume that $|V| = n \le 5.75k \le 6k$.

Idea. Similarly to the polynomial formulation of k-Internal Spanning Tree, we go over all possible $S_1, \ldots, S_m \subseteq V$ such that $\bigcup S_i = V$ and $B(S_1, \ldots, S_m)$ is a tree, and go over all possible k_1, \ldots, k_m such that $\sum k_i \geq k$. For fixed $\{S_i\}$ and $\{k_i\}$, it suffices to check that for every i there exists a spanning tree of $G[S_i]$ with at least k_i leaves, where we should consider $S_i \cap \bigcup_{j \neq i} S_j$ as internal nodes (to do that, we add a leaf to each node from $S_i \cap \bigcup_{j \neq i} S_j$).

Variables. Introduce $s(G, k) = O(\binom{n}{\leq 3n/\theta}) \cdot n^{\theta} \cdot 3n/\theta) = O^*(\binom{6k}{18k/\theta})$ variables:

$$X = \{x_{S,A,k'} \colon A \subseteq S \subseteq V, |S| \le 3n/\theta, |A| \le \theta - 1, k' \le |S| \}.$$

The mapping $\phi(G, k)$ assigns the following value to a variable $x_{S,A,k'}$:

[there exists a spanning tree of $G_{S,A}$ that contains at least k' + |A| leaves].

We add |A| as we obtain a graph with |A| dummy leaves that appear in any spanning tree.

Complexity. The mapping $\phi(G, k)$ can be computed in time $O^*(\binom{6k}{18k/\theta}) \cdot 4^{3n/\theta + \theta}) = O^*(\binom{6k}{18/\theta}) \cdot 4^{20k/\theta}$, since k-Leaf Spanning Tree can be solved in time $O^*(4^k)$ [KLR08].

Polynomial. The polynomial looks as follows:

$$P_{s(G,k)}(X) = \sum_{m \leq \theta} \sum_{\substack{S_1, \dots, S_m \subseteq V, \\ 1 < |S_i| \leq 3n/\theta, \\ \bigcup S_i = V, \\ B(S_1, \dots, S_m) \text{ is a tree}}} \sum_{\substack{k_1, \dots, k_m, \\ k_i \leq |S_i|, \\ \sum k_i \geq k,}} \prod_{i=1}^m x_{S_i, \bigcup_{j \neq i} (S_i \cap S_j), k_i}$$

The number of monomials added to $P_{s(G,k)}$ is $O(\theta \cdot \binom{n}{\leq 3n/\theta})^{\theta} \cdot (3n/\theta)^{\theta}) = \binom{6k}{18k/\theta}^{\theta} k^{O(1)} = 2^{O(k)}$. Degree. The degree of \mathcal{P} is θ .

k-Nonblocker. Given a graph G(V, E), check whether G contains a subset of nodes of size at least k whose complement is a dominating set in G.

Kernel. As there exists a kernel of size $\frac{5k}{3}$ for k-Nonblocker [DFF+06], we assume that $|V| = n \le \frac{5k}{3}$.

Idea. Let $N \subseteq V$ be a nonblocker for G with at least k nodes, and let $D = V \setminus N$ be a dominating set for G. Let $D^1 \sqcup \cdots \sqcup D^{\theta}$ be a partition of D into θ blocks of size at most n/θ . Let $N^1 \sqcup \cdots \sqcup N^{\theta}$ be a partition of N such that each node from N^i is adjacent to some node from D^i , and, for each $i \in [\theta]$, let $N_1^i \sqcup \cdots \sqcup N_{\theta}^i$ be a partition of N^i into θ blocks of size at most n/θ . This way, we get a partition $\coprod N_j^i \sqcup \coprod D^i$ of V into $\theta^2 + \theta$ blocks of size at most $n/\theta \leq \frac{5k}{3\theta}$.

To construct the polynomial $P_{s(G,k)}$, we go through all such partitions, and for each partition we check that for every i each node from $\bigcup_{i} N_{j}^{i}$ is adjacent to a node from D^{i} .

Variables. Introduce $s(G, k) = O(\binom{n}{\leq n/\theta}^2) = O^*(\binom{5k/3}{5k/3\theta}^2)$ variables:

$$X = \{x_{N,D} \colon N, D \subseteq V, |N|, |D| \le n/\theta\}.$$

The mapping $\phi(G, k)$ assigns the following value to a variable $x_{N,D}$:

 $\left[\forall v \in N \ \exists u \in D : \{v,u\} \in E\right] = \left[\text{every node in } N \text{ is dominated by a node in } D\right].$

Complexity. The mapping $\phi(G,k)$ can be computed in time $O^*(\binom{5k/3}{5k/3\theta})^2$.

Polynomial. For every $N \subseteq V$ of size at least k and a partition $\bigsqcup N_j^i \bigsqcup \bigsqcup D^i$ of V into $\theta^2 + \theta$ blocks of size at most n/θ such that $\bigsqcup_{i,j} N_j^i = N$, we add the following monomial to $P_{s(G,k)}$:

$$\prod_{i,j\leq\theta}x_{N_j^i,D^i}.$$

The number of monomials added to $P_{s(G,k)}$ is $O((\theta^2 + \theta)^n) = 2^{O(k)}$.

Degree. The degree of \mathcal{P} is θ^2 .

k-Path Contractibility. Given a graph G, check whether it is possible to contract at most k edges in G to turn it into a path.

Kernel. As there exists a kernel of size 5k + 3 for k-Path Contractibility [HVHL⁺14], we assume that $|V| = n \le 5k + 3 \le 6k$.

Idea. Consider the path resulting from edge contraction. Each node of the path corresponds to a connected set of G that contracts to that node. Let S be a family of those sets of size at least n/θ , and let T be a family of those sets of size less than n/θ . We apply the following procedure to T. While T contains two sets T_1 and T_2 of size less than $n/(2\theta)$ and $T_1 \sqcup T_2$ is a connected set in G, we replace T_1 and T_2 in T by $T_1 \sqcup T_2$.

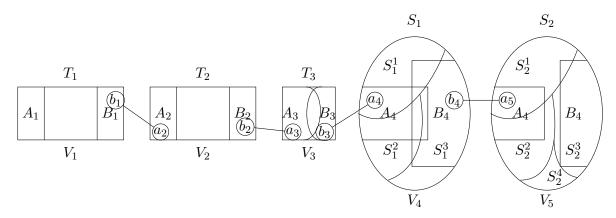
Let $s := |\mathcal{S}|$ and $t := |\mathcal{T}|$. Let $\mathcal{V} = \mathcal{S} \sqcup \mathcal{T}$, and let $\mathcal{V} = \{V_1, \ldots, V_m\}$ where V_i are numbered according to their order in the path. This way, we get a partition $V_1 \sqcup \cdots \sqcup V_m$ of V into m = s + t blocks. In order to obtain an upper bound for m, we consider a partition $\mathcal{T} = \mathcal{T}_\ell \sqcup \mathcal{T}_s$, where \mathcal{T}_ℓ contains sets of size at least $n/(2\theta)$ and \mathcal{T}_s contains sets of size less than $n/(2\theta)$. Observe that $s \leq n/(n/\theta) = \theta$, $|\mathcal{T}_\ell| \leq n/(n/(2\theta)) = 2\theta$, and that after applying the above procedure to \mathcal{T} , for every $V_i \in \mathcal{T}_s$, V_{i-1} and V_{i+1} (if they exist) belong to $\mathcal{S} \sqcup \mathcal{T}_\ell$, so $|\mathcal{T}_s| \leq |\mathcal{S}| + |\mathcal{T}_\ell| + 1 \leq \theta + 2\theta + 1 \leq 4\theta$. Hence, $t = |\mathcal{T}_\ell| + |\mathcal{T}_s| \leq 6\theta$, and $m = s + t \leq 7\theta$.

Let $S = \{S_1, \ldots, S_s\}$ and $T = \{T_1, \ldots, T_t\}$. Sets from S can be too large, so we cover each of them with subsets of size at most $3n/\theta$. For every $i \in [s]$ we apply Lemma 6.1 to a spanning tree of $G[S_i]$ and obtain its subtrees on node sets S_i^1, \ldots, S_i^ℓ , where $\ell \leq \theta$, $\forall j \mid S_i^j \mid \leq 3n/\theta$, $\bigcup_j S_i^j = S_i$, and $B(S_i^1, \ldots, S_i^\ell)$ is connected.

Let k_1, \ldots, k_m be the numbers of contracted edges in V_1, \ldots, V_m , respectively. Let us partition $\{k_i\}$ into $\{k_i^s\}$ and $\{k_i^t\}$, where k_i^s and k_j^t are numbers of contracted edges in S_i and T_j , respectively. Observe that $\forall i \in [s], k_i^s = |S_i| - 1$. That means that $\sum k_i \leq k \Leftrightarrow \sum k_i^t + \sum (|S_i| - 1) \leq k$.

Let A_1, \ldots, A_m and B_1, \ldots, B_m be such sets of nodes that $\forall i \in [m-1]$ B_i consists of nodes from V_i that are adjacent to a node from V_{i+1} , and A_{i+1} consists of nodes from V_{i+1} that are adjacent to a node from V_i . Let us partition $\{A_i\}$ into $\{A_i^s\}$ and $\{A_i^t\}$ such that $\forall i \in [s]$ $A_i^s \subseteq S_i$ and $\forall i \in [t]$ $A_i^t \subseteq T_i$. Similarly, we partition $\{B_i\}$ into $\{B_i^s\}$ and $\{B_i^t\}$.

Consider a_1, \ldots, a_m and b_1, \ldots, b_m such that $\forall i \in [m] \ a_i \in A_i, \ b_i \in B_i$ and $\forall i \in [m-1]$ there is an edge between b_i and a_{i+1} .



To construct the polynomial, we go through all possible $\{V_i\}$ and $\{k_i\}$, and check that every V_i is a connected set, for every i < j there is an edge between V_i and V_j iff i+1=j, and for every V_i of size less than n/θ , the graph $G[V_i]$ can be contracted to a path p_1, \ldots, p_h where only nodes corresponding to p_1 can be adjacent to V_{i-1} and only nodes corresponding to p_h can be adjacent to V_{i+1} . To do that we also go through all possible $\{A_i\}$ and $\{B_i\}$ where $A_i, B_i \subseteq V_i$, $\{a_i\}$ and $\{b_i\}$ where $a_i \in A_i$, $b_i \in B_i$, and $\{S_i^j\}$, and consider $\{T_i\}$, $\{A_i^t\}$ and $\{B_i^t\}$ consistent with $\{V_i\}$, $\{A_i\}$ and $\{B_i\}$.

Variables. Introduce
$$s(G, k) = O(\binom{n}{\leq 3n/\theta}) + \binom{n}{\leq n/\theta}^3 k + \binom{n}{\leq 3n/\theta}^4 + n^2 = O^*(\binom{6k}{18k/\theta}^4)$$
 variables:
$$X = \{x_S \colon S \subseteq V, |S| \leq 3n/\theta\};$$

$$Y = \{y_{T,t}^{A,B} : A, B \subseteq T \subseteq V, |T| \le n/\theta, t \le k\};$$

$$Z = \{z_{V_1,V_2}^{B,A} : V_1, V_2 \subseteq V, B \subseteq V_1, A \subseteq V_2, |V_1|, |V_2| \le 3n/\theta\};$$

$$W = \{w_{b,a} : b, a \in [n]\}.$$

The mapping $\phi(G, k)$ assigns the following values to the variables:

$$\begin{split} x_S \mapsto [G[S] \text{ is connected}] \,; \\ y_{T,t}^{A,B} \mapsto [\text{it is possible to contract at most } t \text{ edges of } G[T] \\ & \text{to obtain a path such that } A \text{ and } B \text{ contract to} \\ & \text{the first and the last nodes in the path, respectively}] \,; \\ z_{V_1,V_2}^{B,A} \mapsto [\text{the set of endpoints of edges between } V_1 \text{ and } V_2 \text{ is a subset of } B \cup A] \,; \\ w_{b,a} \mapsto [b \text{ and } a \text{ are adjacent in } G] \,. \end{split}$$

Complexity. Each variable x_S , $z_{V_1,V_2}^{B,A}$ and $w_{b,a}$ can be computed in polynomial time.

Lemma 6.4. $y_{T,t}^{A,B}$ can be computed in $O^*(2^{|T|})$ time.

Proof. Let p_1, \ldots, p_h be a path obtained from G[T] by contracting edges. Let $T^1 \sqcup \cdots \sqcup T^h$ be a partition of T into sets where T^i contracts to p_i . Let $\tau: T \to \{0,1\}$ be a 2-coloring function that colors $v \in T^i$ to $[i \mod 2]$. We observe that given T and τ , we can obtain the partition $T^1 \sqcup \cdots \sqcup T^h$ by finding connected components of each color.

To compute $y_{T,t}^{A,B}$, we go through all $2^{|T|}$ 2-colorings and check that if we contract the corresponding $\{T^i\}$ we obtain a path, $\sum (|T^i|-1) \le t$, $A \subseteq T^1$ and $B \subseteq T^h$.

The mapping $\phi(G,k)$ can be computed in time $O^*(\binom{6k}{18k/\theta})^4 \cdot 2^{6k/\theta}$.

Polynomial. Before we present the polynomial, we introduce the following monomials.

• $C_{\{S_i^j\}}(X)$ checks that all sets S_i^j are connected. Using it for proper $\{S_i^j\}$, we check that every S_i is connected.

$$C_{\{S_i^j\}}(X) = \prod_{i,j} x_{S_i^j}$$

• $P_{\{T_i\},\{k_i^t\}}^{\{A_i^t\},\{B_i^t\}}(Y)$ checks that, for every i, there is a way to contract at most k_i^t edges of $G[T_i]$ to obtain a path such that A_i^t and B_i^t contract to the first and the last nodes of that path, respectively.

$$P_{\{T_i\},\{k_i^t\}}^{\{A_i^t\},\{B_i^t\}}(Y) = \prod_i y_{T_i,k_i^t}^{A_i^t,B_i^t}$$

• $N_{\{V_i\}}^{\{S_i^j\},\{T_i\}}(Z)$ checks that $\forall a,b$ such that |a-b|>1, there is no edge between V_a and V_b .

$$N_{\{V_i\}}^{\{S_i^j\},\{T_i\}}(Z) = \prod_{\substack{a,b,\\a+1 < b}} \prod_{\substack{M_1,M_2 \in \{T_i\} \cup \{S_i^j\},\\M_1 \subseteq V_a,M_2 \subseteq V_b}} z_{M_1,M_2}^{\varnothing,\varnothing}$$

• $A_{\{V_i\},\{A_i\},\{B_i\}}^{\{S_i^j\},\{T_i\}}(Z)$ checks that for every $i \in [m-1]$ the set of endpoints of edges between V_i and V_{i+1} is a subset of $B_i \cup A_{i+1}$. Using it with the previous monomial, we check that $\forall i \in [t]$, regardless of how we contract $G[T_i]$ to a path, there is no edge connecting an internal node of that path to a node outside T_i .

$$A_{\{V_i\},\{A_i\},\{B_i\}}^{\{S_i^j\},\{T_i\}}(Z) = \prod_{\substack{i \\ M_1,M_2 \in \{T_i\} \cup \{S_i^j\}, \\ M_1 \subset V_i,M_2 \subset V_{i+1}}} z_{M_1,M_2}^{M_1 \cap B_i,M_2 \cap A_{i+1}}$$

• $P_{\{a_i\},\{b_i\}}(W)$ checks that $\forall i \in [m-1]$, there is an edge (b_i,a_{i+1}) between V_i and V_{i+1} .

$$P_{\{a_i\},\{b_i\}}(W) = \prod_i w_{b_i,a_{i+1}}$$

Now, we can present the final polynomial $P_{s(G,k)}(X,Y,Z,W)$: for every partition $V_1 \sqcup \cdots \sqcup V_m$ of V into $m \leq 7\theta$ blocks, $\{A_i\}$, $\{B_i\}$ such that $\forall i \in [m] \ A_i, B_i \subseteq V_i, \{a_i\}$ and $\{b_i\}$ such that $\forall i \in [m] \ a_i \in A_i, \ b_i \in B_i$, the corresponding $\{S_i\}$, $\{T_i\}$, $\{A_i^t\}$ and $\{B_i^t\}$, $\{k_i^t\}$ such that $\sum k_i^t + \sum (|S_i| - 1) \leq k$ and a family $\{S_i^j\}$ of $\ell \leq \theta$ sets of size at most $3n/\theta$ such that $\bigcup S_i^j = S_i$ and $B(S_i^1, \ldots, S_i^\ell)$ is connected, we add the following monomial:

$$C_{\{S_i^j\}}(X) \cdot P_{\{T_i\},\{k_i^t\}}^{\{A_i^t\},\{B_i^t\}}(Y) \cdot N_{\{V_i\}}^{\{S_i^j\},\{T_i\}}(Z) \cdot A_{\{V_i\},\{A_i\},\{B_i\}}^{\{S_i^j\},\{T_i\}}(Z) \cdot P_{\{a_i\},\{b_i\}}(W).$$

The number of monomials added to $P_{s(G,k)}$ is $O((5\theta)^n \cdot 2^n \cdot 2^n \cdot n^{5\theta} \cdot n^{5\theta} \cdot k^{5\theta} \cdot \binom{n}{3n/\theta}^{\theta}) = 2^{O(n)} = 2^{O(k)}$.

Degree. The degree of \mathcal{P} is $\theta^{O(1)}$.

k-Set Splitting. Given a family $\mathcal{F} \subseteq 2^{[n]}$ of size $m = n^{O(1)}$ and an integer k, decide whether there exists a partition $A \sqcup B = [n]$ that splits at least k sets of \mathcal{F} .

Kernel. As there exists a kernel with 2k sets over the universe of size k for k-Set Splitting [LS09], we assume that $|\mathcal{F}| = m \le 2k$ and $n \le k$

Idea. Recall that a partition $A \sqcup B = [n]$ splits a set $S \subseteq [n]$ if there are exists $a, b \in S$ such that $a \in A$ and $b \in B$. For a subfamily $\mathcal{F}' \subseteq \mathcal{F}$, if (A, B) splits exactly q sets of \mathcal{F}' , then there exist two sets $A' \subseteq A$, $B' \subseteq B$ such that $|A'|, |B'| \le q \le |\mathcal{F}'|$ and pair (A', B') splits at least q sets of \mathcal{F}' . Such sets A' and B' can be constructed by picking a pair (a, b) from each split set of \mathcal{F}' . If there are k sets in \mathcal{F} that are split by (A, B), one can partition them into θ subfamilies $\mathcal{F}_1, \ldots, \mathcal{F}_{\theta}$ of size at most k/θ such that (A, B) splits all sets in each \mathcal{F}_i . Then, for each \mathcal{F}_i , one can choose pair of sets $A_i \subseteq A, B_i \subseteq B$ of size at most k/θ that splits all sets in \mathcal{F}_i .

Variables. Introduce $s(G, k) = O(\binom{n}{\leq k/\theta}^2 \binom{m}{\leq k/\theta}) = O^*(\binom{2k}{k/\theta}^3)$ variables:

$$X = \{x_{A,B,L} \colon A,B \subseteq [n], |A|, |B| \le k/\theta, A \cap B = \varnothing, L \subseteq [m], |L| \le k/\theta \} \ .$$

For a set family S of size m and a set of indices $L \subseteq [m]$ denote by S^L a subfamily of S defined by indices of L. The mapping $\phi(\mathcal{F}, k)$ assigns the following value to a variable $x_{A.B.L}$:

$$[(A, B)$$
 splits all sets in \mathcal{F}^L].

Complexity. The mapping $\phi(\mathcal{F}, k)$ can be computed in time $O^*(\binom{2k}{k/\theta})^3$.

Polynomial. For every partition $A \sqcup B = [n]$, sets $A_1, \ldots, A_{\theta} \subseteq A$, $B_1, \ldots, B_{\theta} \subseteq B$ such that $|A_i|, |B_i| \leq k/\theta$ for all i and disjoint sets of indices $L_1, \ldots, L_{\theta} \subseteq [m]$ such that $|L_i| \leq k/\theta$ for all i, we add the following monomial to $P_{s(G,k)}$:

$$\prod_{i \in [\theta]} x_{A_i, B_i, L_i}$$

The number of monomials added to $P_{s(\mathcal{F},k)}$ is at most $O(\binom{n}{k/\theta})^{2\theta} \binom{m}{k/\theta}^{\theta} = 2^{O(k)}$.

Degree. The degree of \mathcal{P} is θ .

Cluster Editing. Given a graph G(V, E) with n nodes and an integer k, decide whether G can be transformed into a cluster graph (i.e., a set of disjoint cliques) using at most k edge modifications (additions and deletions).

Kernel. As there exists a kernel with 2k nodes for Cluster Editing [CM12], we assume that $n \leq 2k$.

Idea. Consider a solution of Cluster Editing for G. Let H be a cluster graph resulting from G by at most k edge modifications. Take all cliques of size at most n/θ and group them into $t \leq 2\theta$ blocks each having at most n/θ nodes. Let T_1, \ldots, T_t be sets of nodes of those blocks. Let S_1, \ldots, S_s be sets of nodes of cliques of size more than n/θ . For every $i \in [s]$, partition S_i into at most θ blocks of size at most n/θ : $S_i = \bigsqcup S_i^j$.

To construct the polynomial, we go through all possible $\{T_i\}$ and $\{S_i^j\}$ and check whether it is possible to modify the graph to obtain a cluster graph that is consistent with $\{T_i\}$ and $\{S_i^j\}$. To do this, let $\mathcal{T} = \{T_i\}$, $\mathcal{S} = \{S_i\}$, $\widetilde{\mathcal{S}} = \bigsqcup_{i,j} \{S_i^j\}$, $\mathcal{V} = \mathcal{T} \sqcup \mathcal{S}$, and $\mathcal{C} = \mathcal{T} \sqcup \widetilde{\mathcal{S}}$. Let

$$\mathcal{V} = \{V_1, \dots, V_{t+s}\}$$
 and $\mathcal{C} = \{C_1, \dots, C_m\}$, where $m \leq 2\theta + \theta^2$.

Let
$$k = \sum_{1 \le a \le b \le m} k_{a,b}$$
, where:

- there are at most $k_{a,a}$ edge modifications in every $C_a \in \mathcal{T}$;
- there are at most $k_{b,b}$ edge additions in every $C_b \in \widetilde{\mathcal{S}}$;
- there are at most $k_{a,b}$ edge additions between every pair of different C_a and C_b , where $C_a, C_b \subseteq S_i$ for some i;
- there are at most $k_{a,b}$ edge deletions between every pair of C_a and C_b , where C_a and C_b belong to different sets from \mathcal{V} .

Now, we can go through all possible \mathcal{T} , \mathcal{S} , \mathcal{V} , and \mathcal{C} that are consistent with each other, and $\{k_{a,b}\}$ such that $\sum k_{a,b} = k$, and check whether for every $a, b \in [m]$ there can be done at most $k_{a,b}$ edge modifications according to cases listed above.

 $Variables. \ \, \text{Introduce} \,\, s(G,k) = O(\binom{n}{\leq n/\theta}^2 \cdot k) = O^*(\binom{2k}{2k/\theta}^2) \,\, \text{variables}:$ $X = \{x_{T,k'} \colon T \subseteq V, |T| \leq n/\theta, k' \in [k]\} \,;$ $Y = \{y_{S,k'} \colon S \subseteq V, |S| \leq n/\theta, k' \in [k]\} \,;$ $Z = \{z_{S_1,S_2,k'} \colon S_1 \sqcup S_2 \subseteq V, S_1, S_2 \leq n/\theta, k' \in [k]\} \,;$ $W = \{w_{C_1,C_2,k'} \colon C_1 \sqcup C_2 \subseteq V, C_1, C_2 \leq n/\theta, k' \in [k]\} \,.$

The mapping $\phi(G, k)$ assigns the following values to the variables:

 $x_{T,k'} \mapsto [\leq k' \text{ edge modifications are needed to make } G[T] \text{ a cluster graph}];$ $y_{S,k'} \mapsto [\leq k' \text{ edge modifications are needed to make } G[S] \text{ a clique}];$ $z_{S_1,S_2,k'} \mapsto [\leq k' \text{ edge modifications are needed to connect each pair of nodes}$ $(v_1,v_2) \text{ by an edge, where } v_1 \in S_1, v_2 \in S_2];$ $w_{C_1,C_2,k'} \mapsto [\leq k' \text{ edge modifications are needed to get rid of all edges } (v_1,v_2) \text{ where } v_1 \in C_1, v_2 \in C_2].$

Complexity. Each variable from $Y \cup Z \cup W$ can be computed in polynomial time. Each variable $x_{T,k'}$ can be computed in time $O^*(3^{|T|})$ by dynamic programming ⁵. The mapping $\phi(G,k)$ can be computed in time $O^*(\binom{2k}{2k/\theta})^2 \cdot 3^{n/\theta}) = O^*(\binom{2k}{2k/\theta})^2 \cdot 3^{2k/\theta})$.

Polynomial. For every $(\mathcal{T}, \mathcal{S}, \widetilde{\mathcal{S}}, \mathcal{V}, \mathcal{C}, \{k_{a,b}\})$, we add the following monomial:

$$\prod_{\substack{a \in [m]: \\ C_a \in \mathcal{T}}} x_{C_a,k_{a,a}} \prod_{\substack{b \in [m]: \\ C_b \in \widetilde{\mathcal{S}}}} y_{C_b,k_{b,b}} \prod_{\substack{a,b \in [m]: \\ a < b, \\ \exists i \in [s]}} z_{C_a,C_b,k_{a,b}} \prod_{\substack{a,b \in [m]: \\ a < b, \\ \exists i,j \in [t+s]: i \neq j, \\ C_a \subseteq V_i,C_b \subseteq V_j}} w_{C_a,C_b,k_{a,b}}.$$

Degree. The degree of \mathcal{P} is $\theta^{O(1)}$.

k-Path. Given a directed graph G(V, E) with n nodes and integer k, decide whether there is a simple path with exactly k nodes.

Idea. Let V = [n]. The color-coding technique, introduced by [AYZ95], solves k-Path as follows: assign a random color $c \in [k]$ to every node; then, all nodes of a k-path receive different colors with probability about e^{-k} ; at the same time, one can find such a colorful path in time $n^{O(1)}2^k$. This gives a randomized $2^{O(k)}n^{O(1)}$ -time algorithm for k-Path. A way to derandomize it is to use a k-perfect family \mathcal{F} of hash functions $f:[n] \to [k]$: go through all $f \in \mathcal{F}$ and for each node $v \in [n]$, assign a color f(v) (see [AYZ95, Section 4]). The key feature of the family \mathcal{F} is: for any subset of nodes $S \subseteq [n]$ of size k, there exists a function $f \in \mathcal{F}$ such that f is injective on S. It guarantees that for any k-path there is a coloring $f \in \mathcal{F}$ that

⁵Let dp[S] be the minimum number of edge modifications that turn G[S] into a cluster graph. Then $dp[S] = \min_{X \subseteq S} (|\overline{E}(X)| + |E(X, S \setminus X)| + dp[S \setminus X])$ and one can compute dp[S] for all $S \subseteq T$ in time $O^*(3^{|T|})$.

assigns different colors to all nodes of the path. [AYZ95] gives a construction of \mathcal{F} of size $O(2^{O(k)} \log n)$. For our purposes, we need a family of smaller size. To achieve this, we allow a larger number of colors.

Take a family \mathcal{F} of $(n, k, \theta k)$ -splitters of size $O(e^{(k/\theta)(1+o(1))}n^{O(1)}) = O^*(e^{2k/\theta})$ that can be computed in time $2^{9k}n^{O(1)}$ guaranteed by Theorem 5.5.

Given a coloring $f: [n] \to [\theta k]$, one can find an f-colorful k-path in time $2^{\theta k} n^{O(1)}$ [AYZ95, Lemma 3.1].

The main idea of the polynomial formulation below is the following. For a coloring $f: [n] \to [\theta k]$, an f-colorful k-path π can be partitioned into θ paths $\pi_1, \ldots, \pi_{\theta}$ such that each path uses at most k/θ colors. Then, a partition $\pi_1, \ldots, \pi_{\theta}$ is valid if the paths are color-disjoint and there is an edge from the last node of π_i to the first node of π_{i+1} , for all $i \in [\theta - 1]$.

Variables. Introduce $s(G, k) = O\left(e^{2k/\theta} {\theta \choose \leq k/\theta} n^{O(1)}\right) = O^*\left(e^{2k/\theta} {\theta \choose k/\theta}\right)$ variables:

$$X = \{x_{f,C,u,v} \colon f \in \mathcal{F}, u, v \in V, C \subseteq [\theta k], |C| \le k/\theta\},$$
$$Y = \{y_{u,v} \colon u, v \in V\}.$$

The mapping $\phi(G, k)$ assigns the following values:

 $x_{f,C,u,v} \mapsto [\text{there is an } f\text{-colorful path from } u \text{ to } v \text{ that uses colors from } C \text{ only}],$

$$y_{u,v} \mapsto [(u,v) \in E]$$
.

Complexity. The mapping $\phi(G, k)$ can be computed in time $O\left(2^{k/\theta}e^{2k/\theta}\binom{\theta k}{\leq k/\theta}n^{O(1)}\right) = O^*\left((2e^2)^{k/\theta}\binom{\theta k}{k/\theta}\right)$.

Polynomial. For every $f \in \mathcal{F}$, disjoint $C_1, \ldots, C_{\theta} \subseteq [\theta k]$ such that $|C_i| \leq \frac{k}{\theta}$, and distinct nodes $v_1, \ldots, v_{2\theta} \in V$, add a monomial:

$$\prod_{i \in [\theta]} x_{f,C_i,v_{2i-1},v_{2i}} \prod_{i \in [\theta-1]} y_{v_{2i},v_{2i+1}}$$

The number of monomials added to $P_{s(G,k)}$ is at most $e^{2k/\theta} {\binom{\theta k}{k/\theta}}^{\theta} n^{O(1)} = 2^{O(k)} n^{O(1)}$.

Degree. The degree of \mathcal{P} is $2\theta - 1$.

k-Tree. Given a tree H on k nodes and a graph G on n nodes, decide whether there is a (not necessarily induced) copy of H in G.

Idea. Let \mathcal{F} be an $(n, k, \theta k)$ -splitter from Theorem 5.5. Let H_1, \ldots, H_m be subtrees of H resulting from applying Lemma 6.1 to H, where $m \leq \theta$ and $|V(H_i)| \leq 3k/\theta$. Let $\mathcal{I} \in V(G)^{V(H)}$ be an isomorphism between H and some subtree of G. Let $T_i = V(H_i)$, and let $S_i = \mathcal{I}(T_i)$. Let C^T be connector nodes of $B(T_1, \ldots, T_m)$, and let C^S be connector nodes of $B(S_1, \ldots, S_m)$. Let $C_i^T = T_i \cap C^T$, let $C_i^S = S_i \cap C^S$, and let $\mathcal{I}_i = \mathcal{I}_{C_i^T}$. Let $f \in \mathcal{F}$ be a coloring of V(G) that assigns different colors to nodes from $\mathcal{I}(V(H))$, and let $F_i = f(S_i)$.

To construct the polynomial, we go through all possible $\{T_i\}$, $\{C_i^S\}$, $\{F_i\}$ and f such that $T_i \subseteq [k]$, $C_i^S \subseteq [n]$, $F_i \subseteq [\theta k]$, $|T_i| = |F_i| \le 3k/\theta$, $|C_i^S| \le \theta - 1$ and $f \in \mathcal{F}$. We check that $\bigcup C_i^S$ equals to the set of connector nodes of $B(C_1^S, \ldots, C_m^S)$, and that $B(T_1, \ldots, T_m)$, $B(C_1^S, \ldots, C_m^S)$ and $B(F_1, \ldots, F_m)$ are isomorphic trees where sets with the same numbers correspond to each other. We compute C^T as connector nodes of $B(T_1, \ldots, T_m)$ and C_i^T as $T_i \cap C^T$. Now, we define \mathcal{I}_i as a bijection between C_i^T and C_i^S obtained from the above isomorphism between $B(T_1, \ldots, T_m)$ and $B(C_1^S, \ldots, C_m^S)$ as a restriction to connector nodes.

Now, we have $(\lbrace T_i \rbrace, \lbrace C_i^T \rbrace, \lbrace C_i^S \rbrace, \lbrace \mathcal{I}_i \rbrace, \lbrace F_i \rbrace, f)$, and we need to check that

- for all i, $H(T_i)$ is a tree;
- for all i, there exists $S_i \subseteq [n]$ and $\mathcal{J} \in \text{Bij}(T_i, S_i)$ such that:

$$-C_i^S \subseteq S_i$$

$$-f(S_i)=F_i$$

- $H[T_i]$ is isomorphic to $G[S_i]$ according to \mathcal{J} ,

$$-\left.\mathcal{J}\right|_{C_i^T}=\mathcal{I}_i.$$

 $Variables. \ \text{Introduce} \ s(H,G,k) = O(\tbinom{k}{\leq 3k/\theta})k^{\theta}n^{\theta}\theta^{\theta}\tbinom{\theta k}{\leq 3k/\theta}e^{2k/\theta}\log n) \ \text{variables}:$

$$X = \{x_T : T \subseteq [k], |T| \le 3k/\theta\};$$

$$Y = \{y_{T,C^T,C^S,\mathcal{I},F,f} : T, C^T \subseteq [k], C^S \subseteq [n], \mathcal{I} \in (C^S)^{C^T}, F \subseteq [\theta k],$$

$$f \in \mathcal{F}, |T| = |F| \le 3k/\theta, |C^T| = |C^S| \le \theta - 1\}.$$

The mapping $\phi(H, G, k)$ assigns the following values to the variables:

$$x_T \mapsto [H[T] \text{ is a tree}];$$

$$y_{T,C^T,C^S,\mathcal{I},F,f} \mapsto [\text{there exists a set } S \subseteq [n] \text{ such that } C^S \subseteq S, \ f(S) = F,$$
and $G[S]$ is a copy of $H[T]$, where nodes of H in C^T
correspond to nodes of G in C^S according to \mathcal{I}].

Complexity. The family \mathcal{F} can be computed in time $2^{9k}n^{O(1)}$. Each variable from X can be computed in polynomial time. Each variable from Y can be computed in time $O^*(8^{3k/\theta})$ by the following dynamic programming algorithm. Let us view H[T] as a rooted tree. Let $T' \subseteq T$ such that H[T'] is connected, let $F' \subseteq F$, and let $v \in [n]$. Let dp[T'][F'][v] = [there exists a subset $S \subseteq [n]$ such that $v \in S$, |S| = |F'|, f(S) = F' and H[T'] is isomorphic to G[S] according to some isomorphism \mathcal{J} which is consistent with \mathcal{I} and which maps the root of H[T'] into v]. For T', let T'' be a subset of T' such that H[T''] and $H[T' \setminus T'']$ are connected, and the root of H[T''] is a son of the root of H[T']. Then,

$$dp[T'][F'][v] = \bigvee_{\substack{F'' \subseteq F', \\ u \in [n] \colon (u,v) \in E(G)}} (dp[T''][F''][u] \wedge dp[T' \setminus T''][F' \setminus F''][v]).$$

Polynomial. For every $(\{T_i\}, \{C_i^T\}, \{C_i^S\}, \{\mathcal{I}_i\}, \{F_i\}, f)$, we add the following monomial:

$$\prod_{i=1}^{m} x_{T_i} \prod_{i=1}^{m} y_{T_i, C_i^T, C_i^S, \mathcal{I}_i, F_i, f}.$$

The number of monomials is at most $\binom{k}{3k/\theta}^{\theta} \binom{k}{\theta}^{\theta} \binom{n}{\theta}^{\theta} (\theta^{\theta})^{\theta} \binom{\theta k}{3k/\theta}^{\theta} e^{2k/\theta} \log n = 2^{O(k)} n^{O(1)}$. Degree. The degree of \mathcal{P} is at most 2θ .

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A Omitted Proofs

A.1 Proof of Theorem 4.4

Theorem 4.4. Let A be a parameterized computational problem with a parameter k. Assume that for every c > 1, there is $\Delta = \Delta(c)$ such that A admits a Δ -polynomial formulation of complexity c^k .

If A is λ^k -SETH-hard for a constant $\lambda > 1$, then at least one of the following circuit lower bounds holds:

- E^NP requires series-parallel Boolean circuits of size $\omega(n)$;
- for every constant $\gamma > 1$, there exists an explicit family of constant-degree polynomials over \mathbb{Z} that requires arithmetic circuits of size $\Omega(n^{\gamma})$.

Proof. Let $\lambda > 1$ be the constant from the theorem statement, $\gamma > 1$ be an arbitrary constant, and $\sigma = \log(\lambda)/(6\gamma)$. Let $\mathcal{I} \times \mathbb{N}$ be the set of all instances of A, where for an instance $(x,k) \in \mathcal{I} \times \mathbb{N}$, k is the value of the parameter. Let \mathcal{P} be a Δ -polynomial formulation of A of complexity $2^{\sigma k}$, for constant $\Delta = \Delta(\sigma) > 0$. We assume that A is λ^k -SETH-hard: there is a function $\delta \colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that for every $q, d \in \mathbb{N}$, q-SAT can be solved in time $2^{(1-\delta)n}$ given a $\lambda^{(1-\varepsilon)k}|x|^d$ -algorithm for A.

If $\mathcal{P}=(P_t)_{t\geq 1}$ does not have arithmetic circuits over \mathbb{Z} of size t^{γ} for infinitely many values of t, then we have an explicit family of constant-degree polynomials that requires arithmetic circuits of size $\Omega(t^{\gamma})$. Hence, in the following we assume that \mathcal{P} has arithmetic circuits over \mathbb{Z} of size ct^{γ} for all values of t for a constant c>0. Under this assumption, we design a non-deterministic algorithm solving q-TAUT in time $2^{(1-\varepsilon)n}$ for every q. This contradicts NSETH and, by Theorem 2.3, implies a super-linear lower bound on the size of series-parallel circuits computing E^{NP} .

Let $\delta_0 = \delta(1/2) \in (0,1)$ where δ is the function from the SETH-hardness reduction for A. Let $\alpha = \frac{1}{(\Delta+1)(\gamma+2\Delta+8)}$, $L = 2^{(1-\delta_0)\alpha n}$, $K = 2(1-\delta_0)\alpha n/\log(\lambda)$, and $T = 2^{(\sigma K)} \cdot L^{\Delta}$. We will start with an instance of the q-TAUT problem on n variables, reduce it to $2^{(1-\alpha)n}$ instances of q-TAUT on αn variables each . Then we'll use the fine-grained reduction from q-SAT to the problem A on instances of length $\ell \leq L$ and parameter $k \leq K$. Finally, we'll use the polynomial formulation of A to reduce instances of length ℓ and parameter k to polynomials with $t \leq T$ variables.

Let F be a k-DNF formula over n variables. In order to solve F, we branch on all but αn variables. This gives us $2^{(1-\alpha)n}$ k-DNF formulas. By solving q-SAT on the negations of all of these formulas, we solve q-TAUT on the original formula F.

Assuming a $\lambda^{k/2}$ -algorithm for A we have a $2^{(1-\delta_0)\alpha n}$ -algorithm for q-SAT on αn variables for $\delta_0 = \delta(1/2) > 0$. We now apply the fine-grained reduction from q-SAT to A to (the negations of) all $2^{(1-\alpha)n}$ instances of q-TAUT. This gives us a number of instances of the problem A. Let ℓ be the largest length of these instances, and k be the largest value of the parameter in these instances. Since the running time of the reduction is bounded from above by $2^{(1-\delta_0)\alpha n}$, we have that $\ell \leq 2^{(1-\delta_0)\alpha n} = L$. From Definition 2.6, we know that $\lambda^{k/2} < 2^{(1-\delta_0)\alpha n}$, so each instance of A has the parameter value at most $k < 2(1-\delta_0)\alpha n/\log(\lambda) = K$.

Since \mathcal{P} is a polynomial formulation of A of complexity $2^{\sigma k}$, there exist $s: \mathcal{I} \times \mathbb{N} \to \mathbb{N}$, $s(x,k) \leq 2^{\sigma k} |x|^{\Delta} \leq 2^{\sigma K} L^{\Delta}$ and $\phi: \mathcal{I} \times \mathbb{N} \to \mathbb{Z}^*$ (both computable in time $2^{\sigma K} L^{\Delta}$) such that for every $x \in \mathcal{I}$ and $k \in \mathbb{N}$,

- $P_{s(x,k)}(\phi(x,k)) \neq 0 \Leftrightarrow (x,k)$ is a yes instance of A;
- $|P_{s(x,k)}(\phi(x,k))| < 2^{2^{\sigma k}|x|^{\Delta}}$.

Using \mathcal{P} , we will solve all instances of A in two stages: in the preprocessing stage (which takes place before all the reductions), we guess efficient arithmetic circuits for polynomials P_t for all $t \leq T$, in the solving stage, we solve all instances of A using the guessed circuits. Note that we'll be using

the polynomials to solve instances of A resulting from q-SAT instances on αn variables. Since L is the largest length of such an instance of A and K is the largest value of the parameter, we have that each such instance is mapped to a polynomial with at most $s(x,k) \leq 2^{\sigma k} |x|^{\Delta} \leq 2^{\sigma K} L^{\Delta} = T$ variables. Therefore, finding efficient arithmetic circuits for polynomials P_t for all $t \leq T$ will be sufficient for solving all q-SAT instances of size αn that we obtain in the reductions.

Preprocessing. For every $t \leq T$, we find a prime p_t in the interval $2^{t+1} \leq p_t \leq 2^{t+2}$ in non-deterministic time $O(t^7)$ [AKS04, LP19].

Now for every $t \leq T$, we reduce all coefficients of the polynomial P_t modulo p_t to obtain a polynomial Q_t over \mathbb{Z}_{p_t} , and let $\mathcal{Q} = (Q_1, Q_2, \ldots)$. For every $t \leq T$, we now non-deterministically solve Gap-MACP_{$\mathcal{Q},\mu\Delta^2,p_t$} (t,ct^{γ}) using Lemma 3.5. Since we assume that \mathcal{P} has arithmetic circuits over \mathbb{Z} of size ct^{γ} , we have that \mathcal{Q} has arithmetic circuits over \mathbb{Z}_{p_t} of this size. Thus, we obtain arithmetic circuits C_t of size at most

$$c\mu\Delta^2t^\gamma$$
 (5)

computing Q_t over \mathbb{Z}_p for all $t \leq T$. Since C_t computes Q_t correctly in \mathbb{Z}_p and $|P_{s(x,k)}(\phi(x,k))| < 2^{s(x,k)} \leq p_{s(x,k)}/2$ for all $x \in I_\ell$, we can use C_t to solve A for every instance length $\ell \leq L$ and parameter $k \leq K$. By Lemma 3.5, Gap-MACP_{$Q,\mu\Delta^2,p_t$} (t,ct^{γ}) can be solved in (non-deterministic) time

$$O\left(\Delta^2 \cdot ct^{\gamma} \cdot t^{2\Delta} \cdot \log^2(p_t)\right) = O\left(T^{\gamma+2\Delta+2}\right).$$

The total (non-deterministic) running time of the preprocessing stage is then bounded from above by the time needed to find T prime numbers, write down the corresponding explicit polynomials modulo p_t , and solve T instances of Gap-MACP:

$$O(T(T^7 + T^{\Delta+2} + T^{\gamma+2\Delta+2})) = O(T^{\gamma+2\Delta+8}) = O(2^{(1-\delta_0)n}),$$
 (6)

where the last equality holds due to $T = 2^{\sigma K} L^{\Delta}$, $L = 2^{(1-\delta_0)\alpha n}$, $K = 2(1-\delta_0)\alpha n/\log(\lambda)$, $\sigma = \log(\lambda)/(6\gamma)$, and $\alpha = \frac{1}{(\Delta+1)(\gamma+2\Delta+8)}$.

Solving. In the solving stage, we solve all $2^{(1-\alpha)n}$ instances of q-SAT by reducing them to A and using efficient circuits found in the preprocessing stage. For an instance x of A of length ℓ with parameter k, we first transform it into an input of the polynomial $y = \phi(x, k) \in \mathbb{Z}^{s(x,k)}$. Both s(x,k) and $\phi(x,k)$ can be computed in time $O(2^{\sigma k}\ell^{\Delta})$. Then we feed it into the circuit $Q_{s(x,k)}$. First we note that we have the circuit $Q_{s(x,k)}$ after the preprocessing stage as $s(x,k) \leq T$ and we have circuits (Q_1,\ldots,Q_T) . The number of arithmetic operations in $\mathbb{Z}_{p_{s(x,k)}}$ required to evaluate the circuit is proportional to the circuit size, and each arithmetic operation takes time $\log^2(p_{s(x,k)}) = O(s(x,k)^2)$. From (5) with $t \leq s(x,k) \leq 2^{\sigma k}\ell^{\Delta}$, we have that we can solve an instance of A with ℓ inputs and parameter k in time

$$O(2^{\sigma k}\ell^{\Delta}) + c\mu\Delta^{2} \cdot s(x,k)^{2} \cdot 2^{\sigma\gamma k} \cdot \ell^{\Delta\gamma} = O\left(2^{3\sigma\gamma k}\ell^{3\Delta\gamma}\right) = O\left(\lambda^{k/2}\ell^{3\Delta\gamma}\right),$$

where the last equality holds due to the choice of $\sigma = \log(\lambda)/(6\gamma)$. The fine-grained reduction from q-SAT to A implies that a $O(\lambda^{k/2}\ell^{O(1)})$ -time algorithm for A gives us a $O(2^{n(1-\delta_0)})$ -time

algorithm for q-SAT. Thus, since we solve each ℓ -instance of A resulting from $2^{(1-\alpha)n}$ instances of q-SAT in time $O(\lambda^{k/2}\ell^{O(1)})$, we solve the original n-variate instance F of q-TAUT in time

$$O\left(2^{(1-\alpha)n}\cdot(2^{\alpha n})^{1-\delta_0}\right) = O\left(2^{n(1-\alpha\delta_0)}\right). \tag{7}$$

The total running time of the preprocessing and solving stages (see (6) and (7)) is bounded from above by $O\left(2^{n(1-\delta_0)}\right) + O\left(2^{n(1-\alpha\delta_0)}\right) = O\left(2^{n(1-\alpha\delta_0)}\right)$, which refutes NSETH, and implies a super-linear lower bound for Boolean series-parallel circuits.

A.2 Proof of Lemma 6.1

Lemma A.1. Let T(V, E) be a rooted tree and $M \subseteq V$ be a set of more than ℓ nodes. Then there is a subtree U of T whose topmost node is u such that $T[V(T) \setminus V(U) \cup \{u\}]$ is connected and $\frac{\ell}{2} \leq |M \cap (V(U) \setminus \{u\})| \leq \ell$.

Proof. We prove this lemma by induction on the size of the tree T. The base case of |V|=1 holds trivially (by choosing U=T). For the case |V|>1, suppose the root r of T has m children with subtrees T_1,\ldots,T_m . Let $a_i=|V(T_i)\cap M|$. Assume without loss of generality that $a_1\leq\ldots\leq a_m$. Consider the following three cases.

- If $a_m > \ell$, then we use the induction hypothesis to find U in T_m .
- If $\frac{\ell}{2} \leq a_m \leq \ell$, then we take U to be the subtree T_m together with the root r. Then $|M \cap (V(U) \setminus \{u\})| = |M \cap V(T_m)| = a_m$.
- If $a_m < \frac{\ell}{2}$, then, for all $i \in [m]$, $a_i < \frac{\ell}{2}$. As $\sum_{i=1}^m a_i \ge |M| 1 \ge \ell$, there exists $j \in [m]$ such that $\frac{\ell}{2} \le \sum_{i=1}^j a_i \le \ell$. We take U to be the subtrees T_1, \ldots, T_j together with the root r. This

way we have $|M \cap (V(U) \setminus \{u\})| = \sum_{i=1}^{j} |M \cap V(T_i)| = \sum_{i=1}^{j} a_i$.

Lemma 6.1. Let T(V, E) be a tree, $M \subseteq V$ be a set of k nodes, and $\theta > 1$ be an integer. Then E can be partitioned into $m \leq \theta$ blocks $E = E_1 \sqcup \cdots \sqcup E_m$ such that, for each $i \in [m]$, E_i induces a subtree T_i of T with at most $\frac{2k}{\theta-1} + 2$ nodes from M.

Proof. Choose an arbitrary node r of T as the root and view T as a rooted tree. Let $\ell = \lceil \frac{2k}{\theta-1} \rceil$. As long as $|V(T) \cap M| > \ell$ we repeat the following procedure. (Note that it's possible that in the very beginning $|V(T) \cap M| = k \le \ell$, so we don't run the procedure even once.) Since $|V(T) \cap M| > \ell$, Lemma A.1 gives us a tree U and its root u. We take $T_i = U$, and delete all the vertices $V(U) \setminus \{u\}$ from T.

When we can't apply this procedure anymore, we have $|V(T)\cap M| \leq \ell$, and we add the remaining tree with at most ℓ nodes from M as the last T_i in our collection. Note that each T_i is a subtree, and each T_i contains at most $\ell+1 \leq \frac{2k}{\theta-1}+2$ nodes from M. Since each application of Lemma A.1 removes at least $\ell/2$ vertices from M, the number of such iterations is at most $\lceil \frac{k}{\ell/2} \rceil = \lceil \frac{2k}{\ell} \rceil$. Thus, the total number of subtrees is $m \leq \lceil \frac{2k}{\ell} \rceil + 1 \leq \lceil \frac{2k(\theta-1)}{2k} \rceil + 1 = \theta$.